MAT 1001 / 458 : Real Analysis II
Assignment 7, due March 15, 2022

1. *Characterization of orthogonal projections (Stein & Shakarchi Problem 4.11*
   Let $\mathcal{H}$ be a Hilbert space.

   (a) The orthogonal projection P onto a closed subspace S is defined by the property that

   $$Px = x \text{ for all } x \in S\,, \qquad Px = 0 \text{ for all } x \in S^\perp .$$

   Verify that P is **idempotent**, i.e., $P^2 = P$, and **self-adjoint**, i.e., $\langle Px, y\rangle = \langle x, Py\rangle$ for all $x, y \in \mathcal{H}$.

   (b) Conversely, every self-adjoint idempotent continuous linear transformation is the orthogonal projection onto some closed subspace, S.

   (c) Also verify that $Q = I - P$ is the complementary projection onto $S^\perp$.

2. *Example of a non-separable Hilbert space (Stein & Shakarchi, Problem 4.2)*
   Consider the collection of exponential functions $f_\lambda(x) = e^{i\lambda x}$ on the real line, where λ ranges over $\mathbb{R}$. Let $\mathcal{H}_0$ denote the space of finite linear combinations of these exponentials. For $f, g \in \mathcal{H}_0$, define an inner product by

   $$\langle f, g\rangle = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(x)\bar{g}(x)\, dx\,.$$

   (a) Show that this limit exists, and compute its value for $f = \sum_{i=1}^n \alpha_i f_{\lambda_i}$, $g = \sum_{j=1}^m \beta_j f_{\lambda_j}$.

   (b) Show that the corresponding norm, given by $||f|| = \sqrt{\langle f, f\rangle}$, satisfies

   $$\sup_x |f(x)| \geq ||f||\,.$$

   (c) Let $\mathcal{H}$ be the completion of $\mathcal{H}_0$ with respect to $||\cdot||$. Prove that $\mathcal{H}$ is not separable.
   Hint: The functions $\{f_\lambda\}_{\lambda\in\mathbb{R}}$ are orthonormal.

3. Describe the Gram-Schmidt-orthogonalization algorithm to a fellow student. Then explain how it can be used to construct an orthonormal basis for a separable Hilbert space.

   (Nothing to hand in)

4. *Orthogonal polynomials*

   (a) Let μ be a measure on an interval such that $L^2(d\mu)$ is infinite-dimensional and contains the space of polynomials as a dense subspace. Describe how to construct an orthogonal basis $(p_n)_{n\geq 0}$ for $L^2(d\mu)$ where each p_n is a polynomial of degree exactly n.

   (b) *Three-term recurrence relation*
   Show that there exist real-valued sequences $(a_n), (b_n), (c_n)$ such that

$$p_{n+1} = (a_n x + b_n)p_n + c_n p_{n-1} \quad (\text{for } n \geq 1)\,.$$

   Hint: Argue that that $\langle xP_n, P_m\rangle = 0$ if $m \leq n-2$.

5. *Gaussian vs. spherical averages*
   A function $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is **homogeneous of degree** α, if

$$f(tx) = t^\alpha f(x) \qquad (t > 0,\ x \in \mathbb{R}^n \setminus \{0\})\,.$$

   Let $\alpha < n$, and assume that f is continuous. For such functions, averages over spheres and balls are related to Gaussian averages by

$$\frac{1}{n\omega_n}\int_{S^{n-1}} f(x)\,d\sigma(x) = \frac{b_n(\alpha)}{\omega_n}\int_{|x|<1} f(x)\,dx = c_n(\alpha)\int_{\mathbb{R}^n} f(x)\,d\gamma(x)\,,$$

   where ω_n is the volume of the unit ball, $b_n(\alpha), c_n(\alpha)$ are constants, and $d\gamma = (2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2}|x|^2}$ is the standard Gaussian probability measure on $\mathbb{R}^n$ (also called the standard multivariate normal distribution).

   (a) Determine the value of the constants $b_n(\alpha)$ and $c_n(\alpha)$. (Use Folland Theorem 2.49)

   (b) Compute $\frac{1}{n\omega_n}\int_{S^{n-1}} |x\cdot v|^4\,d\sigma$, where $v \in S^{n-1}$ is a fixed unit vector.

6. Read the constructive proof of the uniform boundedness principle in Lieb-Loss (Thm. 2.12). You may find it interesting to look more broadly at Sections 2.9-2.13.

   (Nothing to hand in)