MAT 1001 / 458 : Real Analysis II
Assignment 6, due March 1, 2022

1. *Nowhere monotone continuous functions*
   Prove that there exists a continuous function on the unit interval that is not monotone on any subinterval of positive length.

   Hint: Given a closed subinterval $I \subset [0,1]$ of positive length, prove that the set
   $$A_I = \{ f \in \mathcal{C}([0,1]) : f\big|_I \text{ is monotone} \}$$
   is closed and contains no open balls, and then apply the Baire Category Theorem. You may use that $\mathcal{C}([0,1])$, the space of continuous function with the sup norm, is a Banach space, and that the piecewise linear functions form a dense subspace.

2. Let X be an infinite-dimensional Banach space.

   (a) If $V \subset X$ is a finite-dimensional subspace, show that V is closed.

   (b) *Hamel bases.* Let $\mathcal{B} \subset X$ be a maximal set of linearly independent vectors. It is a theorem of Linear Algebra that every element $x \in X$ can be uniquely represented as a finite linear combination
   $$x = \sum_{j=1}^{n} \alpha_j b_j$$
   for some nonnegative integer n, vectors $b_1, \ldots, b_n \in \mathcal{B}$, and coefficients $\alpha_1, \ldots, \alpha_n$. Prove that $\mathcal{B}$ is uncountable.

   Hint: Use the Baire category theorem and Part (a).

3. *Mazur's theorem on strongly convergent convex combinations (see Lieb-Loss Thm. 2.13)*
   Let $(x_n)_{n\geq 1}$ be a sequence in a Banach space X, and $a \in X$. Suppose that $x_n \rightharpoonup a$ (weakly). Prove that there exist coefficients λ_{nj} and a sequence N_n with
   $$\lambda_{nj} \geq 0 \; (1 \leq j \leq N_n) \,, \quad \lambda_{nj} = 0 \; (j > N_n) \,, \quad \sum_j \lambda_{nj} = 1 \quad (\text{for each } n \geq 1) \,,$$
   such that $y_n := \sum_j \lambda_{nj} x_j \;\to\; a$ (strongly).

   Hint: Consider the closed convex hull of $\{x_n : n \geq 1\}$ in X.

4. *L^p-spaces with $0 < p < 1$ (Folland 6.16).* On a measure space (X, μ), let
   $$L^p := \Big\{ f : X \to \mathbb{C} \;\Big|\; \textstyle\int |f|^p \, d\mu < \infty \Big\} \Big/ \mu\text{-a.e.} \,, \qquad (0 < p < 1) \,,$$
   (identifying, as usual, functions that agree μ-almost everywhere). Show that
   $$\rho(f,g) := \int |f-g|^p \, d\mu$$
   defines a metric that makes L^p into a complete topological vector space. (Verify the triangle inequality, the continuity of the vector space operations, and completeness.)

5. *Gamma and Beta.* The Gamma-function is defined by $\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}\,dt$ for $x > 0$. Establish the following properties.

(a) *Functional equation*

$$\Gamma(x+1) = x\Gamma(x) \qquad (x > 0)\,.$$

(b) *Interpolation of the factorials*

$$\Gamma(n) = (n-1)! \qquad (n = 1, 2, \ldots)$$

(c) *Log-convexity*

$$\Gamma\big((1-s)x + sy\big) \le \big(\Gamma(x)\big)^{1-s}\big(\Gamma(y)\big)^s \qquad (x, y > 0;\ s \in (0,1))\,.$$

(d) *The Beta-integral (Folland 2.60)*

$$\mathrm{B}(x,y) := \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt \qquad (x, y > 0)\,.$$

(e) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. (Take $x = y = \frac{1}{2}$ in Part (d), and change variables $t = \sin^2\theta$.)

The *Bohr-Mollerup theorem* says that Γ is uniquely determined by Properties (b) and (c). One consequence:

(f) *Legendre's duplication formula*

$$\Gamma(x) = \frac{2^{x-1}}{\sqrt{\pi}}\Gamma\left(\tfrac{x}{2}\right)\Gamma\left(\tfrac{x+1}{2}\right)\,, \qquad \text{for } x > 0$$

(Check that both sides of the equation are log-concave, satisfy the functional equation, and agree at $x = 1$.)

6. *Polar coordinates on $\mathbb{R}^n$.* Read the construction of the *uniform measure on the unit sphere* in Folland, Section 2.7. The measure of a Borel set $A \subset S^{n-1}$ is defined by

$$\sigma(A) := n\,\mu\big(\{x = ru \in \mathbb{R}^d \mid r \in (0,1), u \in A\}\big)\,,$$

where μ is Lebesgue measure. Note that σ inherits the rotation invariance of μ. The factor n arises from the behavior of Lebesgue measure under dilation (namely, $\mu(rC) = r^n\mu(C)$).

The key formula is Theorem 2.49 of Folland,

$$\int_{\mathbb{R}^n} f(x)\,dx = \int_0^\infty \int_{S^{n-1}} f(ru)r^{n-1}\,d\sigma(u)\,dr\,,$$

which shows that Lebesgue measure equals the product measure $d\mu = r^{n-1}\,dr \times d\sigma$. From here, the measure of the unit sphere and unit ball can be inferred by evaluating the integral of the Gaussian $f(x) = e^{-|x|^2}$ in two ways (by polar coordinates / Fubini). The result is

$$\sigma(S^{n-1}) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} = n\omega_n\,, \qquad \omega_n = \mu(\text{unit ball in } \mathbb{R}^n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}\,.$$

(Nothing to hand-in.)