MAT 1001 / 458 : Real Analysis II
Assignment 4, due February 8, 2022

1. Consider the space ℓ^∞ of bounded sequences $(x_n)_{n\geq 1}$, with the sup-norm. Find the closure of the subspace
$$F := \left\{x \in \ell^\infty \;\middle|\; x_n = 0 \text{ for all but finitely many } n\right\} .$$

2. *Quotient space (Folland 5.1.12).* Let $(E, \|\cdot\|)$ be a normed vector space, and $V \subset E$ a proper closed subspace. Define the quotient space E/V by the equivalence relation
$$x \sim y \;\Leftrightarrow\; x - y \in V .$$
Elements of X/V are equivalence classes $[x] := x + V$.
   (a) Check that
   $$\|x + V\|_{E/V} := \inf_{v\in V} \|x + v\|_X$$
   defines a norm on the quotient space X/V. (Feel free to drop the subscripts.)
   (b) Prove that the canonical projection map $\pi : E \to E/V$ has norm 1.
   (c) If E is complete, then so is E/V. (*Hint:* Use absolutely convergent series.)

3. *Interior of closed convex sets (Brézis Ex. 1.7).* Let C be a closed convex subset of a normed vector space. Assume that C contains a non-empty open subset. Prove that C is the closure of its interior,
$$C = \overline{\operatorname{int} C} .$$
Hint: Argue that for any pair of points $x_0 \in C$, $x_1 \in \operatorname{int} C$, the convex combinations $x_t = (1-t)x_0 + tx_1$ lie in the interior of C for all $t \in (0, 1)$.

4. Recall that a topological space is called **separable** if it contains a countable dense subset.
   (a) Briefly explain why a normed vector space X is separable, if and only if there is a sequence $(x_n)_{n\geq 1}$ in X that spans a dense subspace.
   (b) *X^* separable $\Rightarrow$ X separable (Folland Problem 5.2.25).* If X is a Banach space and its dual space X^* is separable, prove that X is separable. (*Hint:* Let $(f_n)_{n\geq 1}$ be a dense sequence in X^*. For each n, choose x_n with $||x_n|| = 1$ and $|f_n(x_n)| \geq \frac{1}{2}||f_n||$. Then apply Part (a) and a separation argument.)
   (c) *X separable $\not\Rightarrow$ X^* separable.* Show that $L^1(\mathbb{R})$ is separable, but its dual space $L^\infty(\mathbb{R})$ is not. (*Hint:* What is the distance between two indicator functions?)

5. *Krein-Milman theorem (Brézis Problem 1)*
Let X be a normed vector space, and let $K \subset X$ be a non-empty compact convex subset. You will prove that K equals the closed convex hull of its extreme points.

We say that $a \in K$ is an **extreme point** of K, if it cannot be represented as a non-trivial convex combination in K, that is, if for every $x, y \in K$ and $0 < t < 1$

$$(1-t)x + ty = a \quad \Longrightarrow \quad x = y = a\,,$$

More generally, $A \subset K$ is an **extreme subset** if it is closed, non-empty, and for every $x, y \in K$ and $0 < t < 1$

$$(1-t)x + ty \in A \quad \Longrightarrow \quad x, y \in A\,.$$

(a) Consider the collection $\mathcal{E}$ of extreme subsets of K, with the partial order given by containment, i.e., $A \leq B \Leftrightarrow A \supset B$. Prove that $\mathcal{E}$ has a maximal element, A^*.
Hint: Appeal to Cantor's intersection theorem.

(b) *Existence of extreme points*
If A is an extreme subset of K, and $\phi \in X^*$ is a bounded linear functional, show that

$$\left\{ x \in A \;\middle|\; \phi(x) = \max_{y \in A} \phi(y) \right\}$$

is an extreme subset. Conclude that $A^* = \{a\}$ for some extreme point a.
Hint: Hahn-Banach

The **convex hull** of a subset $C \subset X$ consist of all finite convex combinations of points in C,

$$\operatorname{conv} C = \left\{ \sum_{i \in I} t_i x_i \;\middle|\; I \subset \mathbb{N} \text{ finite; } x_i \in C \text{ and } t_i \geq 0\ \forall i \in I;\ \sum_{i \in I} t_i = 1 \right\}.$$

The topological closure $\overline{\operatorname{conv} C}$ is called the **closed convex hull** of C.

(c) Let E be the set of extreme points of K. Prove that $K = \overline{\operatorname{conv} E}$.
Hint: One inclusion is easy; for the other one, use Hahn-Banach.

6. Read the complex Hahn-Banach theorem and its proof (Folland Theorem 5.7). Also read Proposition 5.5, on real vs. complex linear functionals. (Nothing to hand in.)