MAT 1001 / 458 : Real Analysis II
Assignment 3, due February 1, 2022

1. *Tail sum formula.* Let $f \in L^p$, where $1 \le p < \infty$, and let $\lambda_f(t) := \mu(\{x : |f(x)| > t\}$ denote its distribution function.

   Prove that
$$\|f\|_p^p = \int_0^\infty pt^{p-1}\,\lambda_f(t)\,dt\,.$$

2. *The scale of L^p-spaces.* Let (X,μ) be a measure space.
   (a) For $1 \le p < q \le \infty$, show that $L^p \cap L^q$ increases with p (and decreases with q), while $L^p + L^q$ decreases with p (and increases with q).
   (b) Prove that $L^1 \cap L^\infty \subset L^p \subset L^1 + L^\infty$ for all $1 < p < \infty$.
   (c) Furthermore, $L^1 \cap L^\infty$ is dense in L^p for all $1 \le p < \infty$.
   (d) However, $L^1 \cap L^\infty$ is not dense in L^∞ unless μ is finite.

3. *(Best constants in Young's inequality).* In Assignments 1 and 2, you established Young's inequality with constant $C = 1$.
   (a) If $p, q, r > 1$, show that Young's inequality is strict unless one of the three functions f, g, h vanishes a.e.. (*Hint:* Revisit the Hölder estimate from Assignment 1.)
   (b) Brascamp-Lieb [1976] and Beckner [1976] proved that for $p, q, r > 1$, the best-possible constant satisfies $C_{p,q,r,n} < 1$, and is attained by suitable centered Gaussians $f = e^{-a|x|^2}$, $g = e^{-b|x|^2}$ (and $h = e^{-c|x|^2}$). Taking this result for granted, compute the value of the sharp constant $C_{p,q,r,n}$ by optimizing over a, b (and c) > 0.

4. *Additive $\Rightarrow$ linear?* Suppose a function $f : \mathbb{R} \to \mathbb{R}$ satisfies
$$f(x+y) = f(x) + f(y)\,, \quad x, y \in \mathbb{R}\,.$$
   (a) Assuming that f is continuous, prove that it is linear, i.e., there exists $\alpha \in \mathbb{R}$ such that
$$f(x) = \alpha x\,, \qquad x \in \mathbb{R}\,.$$
   Hint: Consider the value of f at rational points.
   (b) Establish the same conclusion under the assumption that f is just measurable.
   Hint: Convolve e^{if} with a smooth bump function ϕ. The convolution is smooth!
   (c) Can you construct a (non-measurable) function f that is additive but not linear? How?

5. *L^p-bound on the maximal operator.* Consider the Hardy-Littlewood maximal operator, defined on locally integrable functions $f : \mathbb{R}^d \to \mathbb{C}$ by

$$Hf(x) := \sup_{r>0} \frac{1}{m((B_r(x))} \int_{B_r(x)} |f(y)|\, dy\,.$$

Here, m denotes Lebesgue measure, and $B_r(x)$ is the ball of radius r centered at x. By the Maximal Theorem (Theorem 3.27 in Folland),

$$m(\{x \in \mathbb{R}^d : Hf(x) > \alpha\}) \le \frac{A}{\alpha}\|f\|_1 \quad \text{for all } \alpha > 0, f \in L^1$$

holds with constant $A = 3^d$. You are asked to find constants C_p for $p > 1$ such that

$$\|Hf\|_p \le C_p \|f\|_p \quad \text{for all } f \in L^p\,. \tag{1}$$

(a) Argue that $\|Hf\|_\infty \le \|f\|_\infty$ for all $f \in L^\infty$, i.e., $C_\infty = 1$.

Let now $f \in L^p$ for some $p \in (1, \infty)$. For $a > 0$ to be chosen below, we split $f \in L^p$ into two layers, $f = g_a + h_a$, where

$$g_a := \operatorname{sign}(f)\, \min\{|f|, a\}\,, \quad h_a := \operatorname{sign}(f)\,(|f| - a)_+\,.$$

We showed in class that g_a is bounded and h_a is integrable, with

$$\|g_a\|_\infty \le a\,, \quad \|h_a\|_1 = \int_a^\infty m(\{x : |f(x)| > t)\, dt\,.$$

(b) *Sublinearity.* Show that for any $a, t > 0$,

$$m(\{|Hf| > 2t\}) \le m(\{|Hg_a| > t\}) + m(\{|Hh_a| > t\})\,.$$

(c) Therefore,

$$m(\{|Hf| > 2a\}) \le \frac{C_1}{a} \int_a^\infty m(\{x : |f(x)| > s|)\, ds\,, \qquad a > 0\,.$$

(d) Insert this estimate into the tail sum formula to bound $\|Hf\|_p^p$ in terms of $\|f\|_p^p$. What value do you obtain for the constant C_p in Eq. (1)?

6. *Real vs. complex interpolation.* In Problem 5, you proved a special case of the Marcinkiewicz interpolation theorem. Read the general statement, Theorem 6.28 in Folland, and compare it with the Riesz-Thorin interpolation theorem (Theorem 6.27). Try find at least 3 important differences. (Nothing to hand in).