MAT 1001 / 458 : Real Analysis II
Assignment 2, due January 25, 2022

1. *Matrix norms.* Recall the definition of the finite-dimensional normed vector spaces ℓ_n^p from Assigment 1, Problem 1. Let $T : \ell_m^p \to \ell_n^p$ be a linear transformation. For $p = 1, 2, \infty$, find the norm of T in terms of its $n \times m$ matrix M.

2. *Uniform smoothness and uniform convexity.* Fix $1 < p \leq 2$.

   (a) Prove that
   $$(a+b)^p + (a-b)^p \geq 2a^p + p(p-1)a^{p-2}b^2\,, \quad \text{for } a > b \geq 0$$
   (By scaling, it suffices to consider the case $a = 1$, $0 \leq b < 1$).

   (b) Recall Hanner's inequality: for all $u, v \in L^p$,
   $$\|u+v\|_p^p + \|u-v\|_p^p \geq \big(\|u\|_p + \|v\|_p\big)^p + \big|\|u\|_p - \|v\|_p\big|^p\,. \tag{1}$$
   Prove the complementary inequality
   $$2^p\big(\|u\|_p^p + \|v\|_p^p\big) \geq \big(\|u+v\|_p + \|u-v\|_p\big)^p + \big|\|u+v\|_p - \|u-v\|_p\big|^p\,. \tag{2}$$

   (c) Suppose $\|u\|_p = \|v\|_p = 1$, and $\|u-v\|_p = \delta > 0$ is very small. Explain how Hanner's inequality implies that the unit ball in L^p (for $1 < p \leq 2$) is uniformly smooth and uniformly convex, i.e., the boundary has no corners and no flat spots.
   (Which of the two inequalities (1) and (2) yields what? Think of the unit ball in ℓ_2^1; a sketch will help.)

   (d) Adapt your argument to the case $2 \leq p < \infty$.

3. *Another proof of Young's inequality (see Assignment 1, Problem 5).*
   Let $p, q, r \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$.

   (a) Show that Young's inequality is equivalent to
   $$\|f * g\|_{r'} \leq C\,\|f\|_p\,\|g\|_q\,,$$
   where r' is the Hölder dual exponent to r (i.e., $\frac{1}{r} + \frac{1}{r'} = 1$).

   (b) Given $p > 1$, prove this inequality with $C = 1$ by interpolating between $q = 1$ and $q = p'$. (Consider the linear transformation $T_f : g \mapsto f * g$.)

4. *Riesz Representation theorem for L^p, $1 \leq p < \infty$.* Read Theorem 6.15 of Folland, its proof, and the subsequent discussion. Compare with the proof presented in the Wednesday's lecture. (Nothing to hand-in)

5. *Differentiability of L^p-norms.* Given $1 < p < \infty$, consider the (nonlinear) functional Ψ on L^p, defined by $\Psi(u) = \|u\|_p^p$.

   (a) *Gâteaux differential.* Prove that the directional derivative

   $$D_v\Psi(u) := \frac{d}{dt}\Psi(u+tv)\Big|_{t=0}$$

   exists for any $u, v \in L^p$.

   (b) Given $u \in L^p$, determine $h \in L^q$ such that $D_v\Psi(u) = \mathbf{Re}\int (hv)\, d\mu\,.$
   Here q is the Hölder dual of p.

   Remark. The map $v \mapsto \int hv\, d\mu$, considered as an element of the dual space $(L^p)^*$, is called the *Gâteaux derivative* of Ψ. This concept of derivative is used extensively in the Calculus of Variations. In general, the Gâteaux differential $D_vF(u)$ may be non-linear and non-continuous in v.

6. Let f, g be nonnegative integrable function on a measure space (X, μ). Prove that

   $$\|f-g\|_1 = \int_0^\infty \mu\big(\{x : f(x) > t\}\ \Delta\ \{x : g(x) > t\}\big)\, dt\,.$$

   Here, $A\Delta B := (A\backslash B) \cup (B\backslash A)$ denotes the *symmetric difference* of two sets A, B.