# MAT 1001 / 458 : Real Analysis II
# Assignment 10, due April 5, 2022

1. Compute the weak (distributional) derivatives of the function $f(x) = [\sin x]_+$ up to order 4.

2. *The fundamental lemma of the Calculus of Variations*
   Prove: If $F \in (C_c^\infty)'(\mathbb{R})$ satisfies $d/d_x\, F = 0$ in the sense of distributions, then $F$ is (represented by) a constant function, i.e., there exists a constant $c$ such that $F(\phi) = c\int \phi$ for all test functions $\phi \in C_c^\infty$.

   *Hint:* First consider the case where $\int \phi = 0$. (Write $\phi = d\Phi/dx$ for some test function $\Phi$.)

3. *(Singularities of $W^{1,p}$-functions)*
   Let $B$ be the unit ball in $\mathbb{R}^n$. For what values of $n \geq 1$, $\lambda > 0$, and $p \in [1, \infty]$ does the function $f(x) = |x|^{-\lambda}$ represent an element of $W^{1,p}(B)$?
   (Start by finding the distributional gradient of $f$).

4. *The Poisson summation formula*

   (a) Let $f \in \mathcal{S}(\mathbb{R})$, and let $\hat{f}$ be its Fourier (integral) transform. Prove that

   $$\sum_{k=-\infty}^{\infty} f(k) = \sum_{k=-\infty}^{\infty} \hat{f}(k)\,.$$

   *Hint:* Consider the Fourier series of the periodic function $F(x) = \sum_{k=-\infty}^{\infty} f(x+k)$.

   (b) Conclude that the **Gaussian sum** $G(t) = \sum_{k=-\infty}^{\infty} e^{-\pi t k^2}$ satisfies $G(t) = t^{-1/2} G(t^{-1})$.

   (c) In particular, $G(t) \sim t^{-1/2}$ as $t \to 0$ (i.e., $\lim_{t\to 0} t^{1/2} G(t) = 1$).

   *Remarks. (a)* The Poisson summation formula also holds in higher dimensions.
   (But the proof in Folland is garbled.) *(b)* The identity for the Gaussian sum extends to the right half-plane $\{\mathrm{Re}\, t > 0\}$ by analytic continuation.

5. *Fourier transform of $|x|^{-\lambda}$ (in the sense of distributions)*
For $\lambda \in (0, n)$, consider the function defined on $\mathbb{R}^n \setminus \{0\}$ by $f_\lambda(x) = |x|^{-\lambda}$.

   (a) Verify that $f_\lambda$ defines a tempered distribution $F_\lambda \in \mathcal{S}'$.

   (b) Suppose you know that its Fourier transform $\hat{F}_\lambda \in \mathcal{S}'$ is also represented by a function (denoted by $\hat{f}_\lambda$). Use rotations and dilations to see that $\hat{f}_\lambda$ must have the form

   $$\hat{f}_\lambda = C f_{n-\lambda}$$

   for some constant $C = C_{n,\lambda}$.

   (c) Argue that the constant $C_{n,\lambda}$ is real and positive.

   (d) Conclude that the *Coulomb energy*

   $$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \phi(x)|x-y|^{-1}\bar{\phi}(y)\,dxdy = C_{3,1}\int_{\mathbb{R}^3} |k|^{-2}|\hat{\phi}(k)|^2\,dk$$

   is real and positive for every $\phi \in \mathcal{S}(\mathbb{R}^3)$.

*Remark.* The functions $f_\lambda$ are called *Riesz-potentials*. See Lieb-Loss Theorem 5.9 for a proper computation of their distributional Fourier transform.

6. *Nyquist's sampling theorem*
Let $f$ be a continuous function on $\mathbb{R}$ such that its Fourier transform satisfies $\hat{f}(k) = 0$ for all $|k| > 1/2$. Such a function is called *band-limited*.

Show that

$$f(x) = \sum_{\ell=-\infty}^{\infty} \frac{\sin(\pi(x-\ell))}{\pi(x-\ell)} f(\ell)\,,$$

that is, $f$ is completely determined by its values at the integers.

*Remark.* The right hand side is evaluated by *"sampling"* the value of the *"signal"* $f$ at the points $\pi\ell$, where $\in \mathbb{Z}$. The formula then *"reconstructs"* $f$ on the entire real line. The corresponding result for a bandwidth of $L \neq 1/2$ is obtained by scaling (here, the function needs to be sampled at intervals of length $\pi/L$.) For the human ear, $L \approx 22,000$Hz, a key parameter in digital recording and compression.

**(Nothing to hand-in for Problem 6)**