MAT 1001 / 458 : Real Analysis II
Assignment 1, due January 18, 2022

1. *The spaces ℓ^p_d.* Consider the functions on $\mathbb{R}^d$ defined by

$$\|x\|_p := \begin{cases} \left(\sum_{i=1}^d |x_i|^p\right)^{\frac{1}{p}}, & \text{for } 0 < p < \infty, \\ \max_{i=1,\dots d} |x_i|, & \text{for } p = \infty\,. \end{cases}$$

   (a) Briefly explain why this defines a norm when $p \geq 1$ but not for $p < 1$.

   (b) Sketch the unit balls for $p \geq 1$ in dimension $d = 2$.
   (Use one picture, and include at least the values $p = 1, 2, \infty$).

   (c) *Equivalence of norms.* For $1 < p < q < \infty$, show that

$$\|x\|_\infty \leq \|x\|_q \leq \|x\|_p \leq \|x\|_1 \leq d\|x\|_\infty\,, \qquad \text{for all } x \in \mathbb{R}^d\,.$$

   (d) Conclude that all these norms define the same topology on $\mathbb{R}^d$.

2. In a few words, can you tell me where you are in your studies?
   Do you know where your interests lie?
   What do you hope to gain from this course?

3. *(Folland 6.5)* Let $(X, \mathcal{M}, \mu)$ be a measure space, and $1 \leq p < q < \infty$. Show that ...

   (a) ... $L^p \subset L^q$ if and only if X does not contain sets of arbitrarily small positive measure;

   (b) ... $L^q \subset L^p$ if and only if X does not contain sets of arbitrarily large finite measure.

   Additional question (not to be handed in): What about the case $q = \infty$?

4. *(Folland 6.38)* Let (X, μ) be a measure space, and $1 \leq p < \infty$. Show that

$$f \in L^p(d\mu) \quad \Longleftrightarrow \quad \sum_{k=-\infty}^{\infty} 2^{kp}\mu(\{x : |f(x)| > 2^k\}) < \infty\,.$$

5. *Young's inequality* says that, for suitable values of p, q, r, n, there is a constant C such that

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x)g(x-y)h(y)\,dxdy \right| \leq C\,\|f\|_p\,\|g\|_q\,\|h\|_r$$

for all $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, and $h \in L^r(\mathbb{R}^n)$.

(a) *Dilation.* For $\lambda > 0$, define $f_\lambda(x) = f(\lambda^{-1}x)$, and correspondingly for g and h. Assuming Young's inequality holds for some p, q, r, n, derive a necessary condition on these parameters.

(b) Prove Young's inequality by applying Hölder's inequality to the functions

$$\begin{aligned} \alpha(x,y) &= |g(x-y)|^{q/p'}|h(y)|^{r/p'}\,, \\ \beta(x,y) &= |h(y)|^{r/q'}|f(x)|^{p/q'}\,, \\ \gamma(x,y) &= |f(x)|^{p/r'}|g(x-y)|^{q/r'}\,. \end{aligned}$$

6. *The bathtub principle.* Let V be a real-valued (Lebesgue-) measurable function on a domain $\Omega \subset \mathbb{R}^d$ such that the sub-level sets $S_t := \{x : f(x) < t\}$ have finite measure for each $t \in \mathbb{R}$. Given $M > 0$, consider the problem of minimizing

$$I(g) = \int V(x)g(x)\,dx$$

among all functions g with $0 \leq g \leq 1$ and $\int g = M$.

(a) Prove that the minimum is assumed by the characteristic function of some measurable set $A \subset \Omega$.

(b) Describe all possible minimizers. Under what conditions on M is the minimizer unique (up to a set of measure zero)?

Hint: Try $A = S_t$ for a suitable choice of t.