Moreover, a simple description of a bijection between A_n and B_n is given by the map $\pi \mapsto (\sigma_\pi, Y_\pi)$.

ACKNOWLEDGMENTS. I thank the referee for a careful reading and several constructive suggestions, including the explicit description of the pairing σ_π in the last section, and Victor Moll for getting me interested in combinatorics.

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Iterated Products of Projections in Hilbert Space

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1. INTRODUCTION. The theorem on convergence of the iterated product of orthogonal projections in Hilbert space has an interesting history. As pointed out by Deutsch [**6**], convergence was first established for two projections by von Neumann [**13**] in 1933, and this result was rediscovered independently by Aronszajn [**2**] in 1950, Nakano [**11**] in 1953, and Wiener [**14**] in 1955. The first proof for an arbitrary finite number of projections was given by Halperin [**7**] in 1962. The result is the following:

Theorem 1. *Let P_j $(1 \le j \le r)$ be the orthogonal projection onto the closed subspace M_j of the Hilbert space $\mathcal{H}$, and let P_M be the orthogonal projection onto the intersection $M = M_1 \cap \cdots \cap M_r$. If $T = P_r \cdots P_1$, then $T^k \to P_M$ strongly as $k \to \infty$, that is, $\|T^k x - P_M x\| \to 0$ for each x in $\mathcal{H}$.*

An elementary proof in the case $r = 2$ appeared recently in [**4**]. Short proofs of a more general result (which we cover later) appeared in [**1**] and [**3**]. An important special case is the iterative procedure of Kaczmarz [**8**] for solving large linear systems. In this light the result provides a theoretical basis for one of the early algorithms in computed tomography [**12**].

We were curious as to why so many well-known mathematicians came upon this result and how they proved it. Nakano's book [**11**] contains a proof in the case $r = 2$, but no application or reference to Theorem 1 save a citation to his 1940 paper [**10**] (written in Japanese). With hopes that we could interpolate, if not translate, Nakano's paper, we requested a copy. (Through the intercession of St. Jerome) we also obtained a related paper by Nakano's student S. Kakutani [**9**]. That paper is central to this note.

After proving Theorem 1 in the case of two projections, Kakutani states that his proof can be extended to show weak convergence in the case of an arbitrary finite number of projections, but he is not sure if the result is true for strong convergence. In filling in the details of Kakutani's remark, it became clear that only an elementary, indeed trivial, observation is needed to complete a proof of strong convergence in the general case. Due to its elementary proof and many applications (see [**6**]), Theorem 1 should be more widely known.

The Nakano [**10**] and Kakutani [**9**] papers are a visual delight. They are dated the "15th year Shyowa era 1/29" (January 29, 1940) and handwritten in Japanese, with many mathematical terms written in cursive English (e.g., Hermitian operator, strongly, weakly compact, projection.) One term is in German. The papers are indeed easy to interpolate.

2. KAKUTANI'S LEMMA AND PROOF OF THEOREM 1. For the proof of Theorem 1, all one needs to know about an orthogonal projection P is that P is linear, idempotent ($P^2 = P$), and self-adjoint ($P = P^*$, its adjoint; i.e., $\langle Px, y\rangle = \langle x, Py\rangle$ for all x and y in $\mathcal{H}$) and that $\|Px\| \le \|x\|$, with equality if and only if $Px = x$. A consequence of the latter fact is that $Tx = P_r \cdots P_1 x = x$ if and only if x belongs to M. Since $T^* = P_1 \cdots P_r$, the same holds for T^*. The following lemma is essentially due to Kakutani [9, p. 43]:

Lemma. *For each x in $\mathcal{H}$, $\|T^k x - T^{k+1}x\| \to 0$ as $k \to \infty$.*

Proof. Let x be in $\mathcal{H}$. Since $\{\|T^k x\|\}$ is a decreasing sequence that is bounded below, it converges. In particular,

$$\|T^k x\|^2 - \|T^{k+1}x\|^2 \to 0 \tag{1}$$

as $k \to \infty$. The Pythagorean theorem gives for any orthogonal projection P and any x in $\mathcal{H}$

$$\|x - Px\|^2 = \|x\|^2 - \|Px\|^2. \tag{2}$$

Let $Q_0 = I$ and for $j = 1, \dots, r$ recursively define $Q_j = P_j Q_{j-1}$, so that $Q_r = T$. The triangle inequality, the inequality $2|a|\,|b| \le a^2 + b^2$, and (2) with $Q_j T^k x$ in place of x lead to

$$\begin{aligned}
\|T^k x - T^{k+1}x\|^2 &= \left\|\sum_{j=0}^{r-1}(Q_j T^k x - Q_{j+1}T^k x)\right\|^2 \le \left[\sum_{j=0}^{r-1}\|Q_j T^k x - Q_{j+1}T^k x\|\right]^2 \\
&\le r\sum_{j=0}^{r-1}\|Q_j T^k x - Q_{j+1}T^k x\|^2 = r\sum_{j=0}^{r-1}(\|Q_j T^k x\|^2 - \|Q_{j+1}T^k x\|^2) \\
&= r(\|Q_0 T^k x\|^2 - \|Q_r T^k x\|^2) = r(\|T^k x\|^2 - \|T^{k+1}x\|^2).
\end{aligned}$$

The lemma follows immediately from (1). ■

Proof of Theorem 1. Write $T^k - T^{k+1} = T^k(I - T)$. From Kakutani's lemma and continuity it follows that $T^k y \to 0$ strongly when y lies in $\overline{\text{range}(I - T)}$, where the overbar denotes closure. But

$$\mathcal{H} = \overline{\text{range}(I - T)} \oplus (\text{range}(I - T))^{\perp} = \overline{\text{range}(I - T)} \oplus \text{null}(I - T^*).$$

Because $T^\star = P_1 \cdots P_r$, $\text{null}(I - T^\star) = M_1 \cap \cdots \cap M_r = M$. Thus, for x in $\mathcal{H}$ we have the orthogonal decomposition $x = y + z$ with y in $\overline{\text{range}(I - T)}$ and z in M. We conclude that $T^k x = T^k y + T^k z = T^k y + z \to z$ strongly as $k \to \infty$. ■

3. AN EXTENSION. A short proof of an extension of Theorem 1 was given by Amemiya and Ando [**1**] and independently by Bauschke, Deutsch, Hundal, and Park [**3**]. In this result the orthogonal projections P_i are all replaced with linear operators A_i that are nonexpansive ($\|A_i\| \le 1$) and nonnegative (self-adjoint and $\langle A_i x, x\rangle \ge 0$ for each x in $\mathcal{H}$). All that is needed for the proof of Kakutani's lemma to be valid in this case is to replace the Pythagorean theorem (2) with the inequality

$$\|x - Ax\|^2 \le \|x\|^2 - \|Ax\|^2 \tag{3}$$

whenever A is nonexpansive and nonnegative. To this end note that

$$\|x - Ax\|^2 = \|x\|^2 - \|Ax\|^2 - 2\langle A(I - A)x, x\rangle,$$

so we need to establish that

$$\langle A(I - A)x, x\rangle \ge 0 \qquad (x \in \mathcal{H}). \tag{4}$$

It is easy to see that A^j is nonexpansive and nonnegative for all nonnegative integers j. Now suppose that $\|A\| < 1$. Then $I - A$ is invertible with inverse given by a uniformly convergent Neumann series $(I - A)^{-1} = \sum_{j=0}^{\infty} A^j$. Moreover, $A(I - A)^{-1} = \sum_{j=1}^{\infty} A^j$ is clearly nonnegative. Writing $x = (I - A)^{-1} y$ we obtain

$$\langle A(I - A)x, x\rangle = \langle Ay, (I - A)^{-1} y\rangle = \langle y, A(I - A)^{-1} y\rangle \ge 0.$$

Hence (4) holds when $\|A\| < 1$. If $\|A\| = 1$ apply this fact to αA with $0 < \alpha < 1$. By continuity in α,

$$0 \le \lim_{\alpha \to 1-} \langle \alpha A(I - \alpha A)x, x\rangle = \langle A(I - A)x, x\rangle.$$

Thus (4) holds in this case as well.

Theorem 2. *Let A_j $(1 \le j \le r)$ be a nonexpansive, nonnegative operator on the Hilbert space $\mathcal{H}$, and let $T = A_r \cdots A_1$. If M is the null space of $I - T$ and P_M is the orthogonal projection on M, then $T^k \to P_M$ strongly.*

Proof. Note that by (3) $\|A_j x\| = \|x\|$ if and only if $A_j x = x$. Consequently, this also holds for T. Since T^* is the product in the reverse order, $Tx = x$ if and only if $T^* x = x$. So, $\text{null}(I - T^*) = \text{null}(I - T) = \cap_{j=1}^{r} \text{null}(I - A_j)$. The proof of Theorem 2 is then completed exactly as that of Theorem 1. ■

The proofs of Theorem 2 given in [**1**] and [**3**] are as short, but less elementary than the one given here. Both use standard, but not completely trivial, results about nonnegative operators to establish (3). The inequality (3) is "condition (φ)" of Halperin [**7**] with $\varphi(t) = t^2$. Once (3) is established, Theorem 2 is a special case of [**7**, Theorem 2]. Also, when the more general setting in Halperin is reduced to that of Theorem 1, his proof is essentially the one given here.

4. REMARKS. Probably Kakutani did not find the foregoing proof for two reasons. First, the topic was a sidelight in an illustrious career, and it is doubtful that he spent much time on it. Second, his approach in the two-projection case led him in a different direction. In [**9**] Kakutani shows that it suffices to consider the case where $M = \{0\}$, after which he establishes the lemma for an arbitrary weakly convergent subsequence $\{T^{k_\nu}x\}$, say $T^{k_\nu}x \to y$ weakly. The lemma is used to show that y then lies in $M = M_1 \cap M_2$, so $y = 0$. Then he uses the weak convergence and a clever manipulation of the two projections to establish strong convergence. With the exception of the last step, everything goes through for an arbitrary finite number of projections.

Kakutani's work on the problem of iterated products of projections was not completely unknown heretofore. In a note added in proof, Halperin [**7**, p. 99] states "Felix Browder has kindly informed me that part of the argument of this paper was used previously by S. Kakutani to obtain a weaker form" of Theorem 1 and refers the reader to [**5**]. Reference [**5**] contains no citation to a paper by Kakutani, but Browder states there that he is "indebted to S. Kakutani for making him acquainted with the latter's unpublished results on the convergence of iterated projections in Hilbert space" [**5**, p. 69]. Browder's paper contains a slight extension of Kakutani's result. It establishes weak convergence in a setting allowing for an infinite number of different projections.

Nakano [**10**] (in a proof different from that in [**11**]) applies the spectral theorem for bounded self-adjoint operators to the operator $(P_1P_2P_1)^k$. His approach generalizes to establish that $(P_1 \cdots P_r \cdots P_1)^k$ converges strongly to P_M, but it is not clear how to get convergence of $(P_r \cdots P_1)^k$ from this when $r > 2$. In the introduction to [**7**], Halperin remarks that the proofs in [**13**], [**11**], and [**14**] can be extended to show strong convergence for any finite number of projections in the symmetric case $T = P_1 \cdots P_r \cdots P_1$. An elementary proof of this latter result appears in [**3**].

Why did so many eminent mathematicians discover this result? The answer: because it is a natural question in Hilbert space. At least that appears to be the case for von Neumann [**13**], Nakano [**10**], [**11**], and Kakutani [**9**]. Aronszajn [**2**] uses the result to obtain a formula for the orthogonal projection on the sum $M_1 + M_2$ in terms of the orthogonal projections P_1 and P_2, and applies this to obtain a formula for the reproducing kernel for the sum of two closed subspaces in terms of the reproducing kernels for the summands. Wiener [**14**] used the result in determining necessary and sufficient conditions for a Hermitian matrix $H(\theta)$ $(-\pi < \theta < \pi)$ to be factorable in the form $H(\theta) = M(\theta)M^*(\theta)$ with M in L^2.

ACKNOWLEGMENTS. We thank the reviewers for pointing out the results in [**3**] and [**4**] and for other helpful remarks. We thank Shun Tanaka for translating portions of [**9**] and [**10**]. Lastly, we thank the librarian at Osaka who provided copies of both [**9**] and [**10**] in response to our request for [**10**].

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Noting the Difference: Musical Scales and Permutations

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Clough and Myerson's paper "Musical Scales and the Generalized Circle of Fifths" [**2**] and its companion paper in the *Journal of Music Theory* [**1**] are important and influential articles in the mathematical music theory literature. They deal with "the way the diatonic set (the white keys on the piano) is embedded in the chromatic scale (all the keys on the piano)" [**2**, p. 695] and give beautifully clear proofs of some elegant theorems. However, there is a slight error in some of the early results and proofs, which we point out and correct in this note. (For more background and detailed analysis of the material that follows, we highly recommend the original papers. We will follow the notation in [**2**] as closely as possible.)

Consider the two tone progressions A-B-C and C-D-E. They seem to have the same structure—each note is followed by the next note in the diatonic. However, if we play the two progressions, they sound quite different. This is because B and C are only one semitone apart (i.e., there is no black key between them on the keyboard; see Figure 1), whereas D and E, the corresponding pair in the second progression, are two semitones apart (there is a black key between them). Roughly stated, the question that Clough

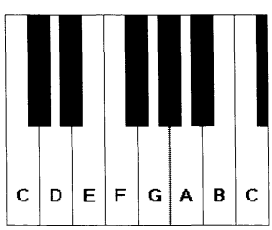

Figure 1. A piano keyboard.