MAT 257: Analysis II
Assignment 8, February 27 2008

Summary

We have constructed the **differential** $d : \Omega^k(U) \to \Omega^{k+1}(U)$ as the unique linear transformation that (a) agrees with the ordinary derivative for $k = 0$, (b) satisfies the Leibniz rule, and (c) satisfies $d^2 = 0$. Another useful property of d is that for any smooth function $g : U \to V \subset \mathbb{R}^m$ and every $\omega \in \Omega^k(V)$,

$$d(g^*\omega) = g^*(d\omega)$$

in $\Omega^k(U)$. This implies that the operator d does not depend on the choice of variables.

- We say that a differential k-form ω is **closed**, if $d\omega = 0$.
- The differential form ω is **exact**, if there exists a $k-1$-form η such that $\omega = d\eta$.

By construction of d, every exact form is closed. The **Poincaré lemma** says that, conversely, every closed form on a **star-shaped** open set is also exact, i.e., given $\omega \in \Omega^k(U)$ with $\omega = 0$, we can find $\eta \in \Omega^{k-1}(U)$ such that $\omega = d\eta$. Note that constructing η amounts to solving a partial differential equation! The assumption that U is star-shaped can be relaxed – the Poincaré lemma clearly remains valid on any set that can be mapped to a star-shaped set by a diffeomorphism. The precise topological condition is that U should be **simply connected**, i.e., it should have no 'holes'.

Let $U \subset \mathbb{R}^k$ be an open set, let $\alpha : U \to \mathbb{R}^n$ be a smooth function, and let ω be a k-form on $\mathbb{R}^n$. We define the integral of ω over the k-dimensional parametrized surface S_α as

$$\int_{S_\alpha} \omega = \int_U \alpha^*\omega\big(e_1, \ldots, e_k\big) = \int_U \omega(\alpha(t))\big(\alpha'(t)e_1, \ldots, \alpha'(t)e_k\big)\ .$$

Here, the last two integrals are ordinary Riemann integrals, and $t = (t^1, \ldots, t_k)^t$ denotes the variable in U. It is not hard to check that this definition is invariant under orientation-preserving reparametrization: If $\beta : V \to U$ is a diffeomorphism, then the Change of Variables formula implies that

$$\int_{S_{\alpha\circ\beta}} \omega = \int_{S_\alpha} \omega\ ,$$

provided that $\det \beta'(z) > 0$ for all $z \in V$; if $\beta'(z) < 0$ on V the sign is reversed.[1]

Assignments:

Read Sections 2 and 3 of Chapter 4 and work the problems up to 4-22.

1. *[Problem 4-14 in Spivak / Problem 1 in Section 29 of Munkres].*
   Let $c : [0, 1] \to \mathbb{R}^n$ be a differentiable curve in $\mathbb{R}^n$. Define the **velocity vector** of c at time t by $v(t) = \big(c'(t)\big)_{c(t)}$.

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[1] Cf. the integral of a scalar function over S_α.

(a) Check that $v(t) = c_*\big((e_1)_t\big)$.

(b) Given an arbitrary vector $w_p = (p, w) \in \mathbb{R}^n_p$, find a curve c such that $c(0) = p$ and w_p is the velocity vector of c at time $t = 0$. (In fact, there are many such curves). Thus, $\mathbb{R}^n_p$ can be viewed as the space of all possible velocity vectors of curves passing through p (hence the name 'tangent space').

(c) If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable, prove that the velocity vector of $f \circ c$ at t is $f_*\big(v(t)\big)$.

2. [*An elaborate version of Spivak's Problem 3-34; Problem 4 of November 28.*]
   We say a function $F : \mathbb{R}^n \to \mathbb{R}^n$ is an **exact vector field** if there exists a differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ such that $F = \nabla V$. The function V a called a **potential** for F.

   (a) If F is a continuously differentiable exact vector field on $\mathbb{R}^n$, prove that necessarily

   $$D_i F^j(x^1, \ldots, x^n) = D_j F^i(x^1, \ldots, x^n)$$

   for all $i \neq j$ and all $x \in \mathbb{R}^n$.

   (b) Conversely, if F is continuously differentiable and its partial derivatives satisfy the condition in (a), show by direct computation that the function V defined by

   $$V(x^1, \ldots, x^n) = \sum_{j=1}^{n} \int_0^{x^j} F^j(x^1, \ldots, x^{j-1}, t, 0, \ldots, 0)\, dt$$

   is a potential for F.

   (c) If F is an exact vector field and V is a potential for F, then, for any smooth curve S_γ given by $t \mapsto \gamma(t)$ with endpoints $\gamma(t_1) = x_1$ and $\gamma(t_2) = x_2$, show that

   $$\int_{S_\gamma} \langle F, \tau \rangle = V(x_2) - V(x_1)\,,$$

   where $\tau(t) = \gamma'(t)/|\gamma'(t)|$ is the unit tangent to the curve at the point $\gamma(t)$. (Assume that $\gamma'(t) \neq 0$ for all t.) Conclude that the potential for a given exact vector field is determined uniquely up to an additive constant.

3. *[Problem 4-21 in Spivak / Problem 5 in Section 30 of Munkres].*
   Define a 1-form on $\mathbb{R}^2 \setminus \{0\}$ by

   $$\omega = \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy\ .$$

   (a) Prove that ω is **closed**, i.e., $d\omega = 0$.

   (b) Let U consist of $\mathbb{R}^2$ with the negative x-axis removed. Construct a function θ on U such that $\omega = d\theta$.
   Hint: In the positive quadrant, try $\theta = \tan^{-1}(y/x)$.

   (c) Argue that the requirement $\omega = d\theta$ determines θ uniquely up to a constant of integration, and use this to show that ω is **not exact** in $\mathbb{R}^n - \{0\}$.

4. Discuss Problems 2 and 3 in light of the Poincaré lemma.