

APM 351: Differential Equations in Mathematical Physics

Test 2, March 16, 2011

(Five problems; 20 points each. Time: 2 hours.)

1. (a) Assume that f is a smooth function that minimizes the Rayleigh quotient

$$\frac{\int_{-1}^1 (1+x^2) |f'(x)|^2 dx}{\int_{-1}^1 |f(x)|^2 dx}$$

among *all* smooth functions on $[-1, 1]$. Write down the **Sturm-Liouville** eigenvalue problem that f solves.

(b) Let λ, μ be two different eigenvalues of this Sturm-Liouville problem, and let f, g be the corresponding eigenfunctions. Prove that f and g are *orthogonal*.

(c) It is known that the eigenfunction corresponding to the lowest eigenvalue of a Sturm-Liouville problem is always strictly positive. Prove that the lowest eigenvalue is *simple*, i.e., the corresponding eigenspace is one-dimensional.

(d) Conclude that the eigenfunction corresponding to the lowest eigenvalue is *even*.

2. (a) What are the defining properties of a **Green's function** for a domain $U \subset \mathbb{R}^2$?

(b) Let $G(x, y)$ be the Green's function for U . How can G be used to solve Laplace's equation

$$\begin{cases} -\Delta u = 0, & \text{in } U, \\ u = g, & \text{on } \partial U? \end{cases}$$

(c) Construct the Green's function for the Laplacian on the complement of the *unit disc*

$$\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 > 1\}.$$

Please justify why your construction yields the required properties!

3. (a) State the strong maximum principle for the **heat equation** $u_t = \Delta u$ on \mathbb{R}^d .

(b) Use the maximum principle to show that if the initial-value problem

$$\begin{aligned} u_t &= \Delta u, & (x \in U) \\ u(x, t) &= 0, & (x \in \partial U) \\ u(x, 0) &= g(x) \end{aligned}$$

has a solution, then it must be *unique*.

(c) Let f be a bounded function on \mathbb{R} with compact support. Verify that

$$u(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{|x-y|^2}{4t}} f(y) dy$$

solves the heat equation for $t > 0$ and $x \in \mathbb{R}$.

(d) If f is continuous, except for a single jump at $x = a$, prove that

$$\lim_{t \rightarrow 0} u(a, t) = \frac{1}{2} \{f(a^-) + f(a^+)\}.$$

Here, $f(a^-)$ and $f(a^+)$ are the limits of f as $x \rightarrow a$ from the left and right hand side, respectively.

4. Consider a solution u of the **wave equation**

$$\begin{cases} u_{tt} = \Delta u, & x \in \mathbb{R}^n, t > 0 \\ u(x, 0) = \phi(x), u_t(x, 0) = 0, & x \in \mathbb{R}^n. \end{cases}$$

Assume that ϕ is smooth, bounded, and vanishes for $|x| > 1$.

(a) In dimension $n = 1, 2, 3$, where does u have to vanish? (A sketch would be useful.)

(b) State *Huygens' principle*.

5. [(20pts)] (a) Construct a basis for the **spherical harmonics** (harmonic polynomials) of degree four in three variables (x, y, z) . Briefly explain your method.

(b) What is the dimension of the space of *all* homogenous polynomials of degree 4 in x, y, z ?

(c) Argue by dimension-counting that every homogeneous polynomial of degree four can be written as

$$P = P_4 + (x^2 + y^2 + z^2)P_2 + (x^2 + y^2 + z^2)^2,$$

where P_4 and P_2 are harmonic polynomials.

(d) Use this to solve

$$\begin{aligned} \Delta u &= 0, & (x^2 + y^2 + z^2 < 1) \\ u &= (2x^2 + y^2)(x^2 + y^2 + z^2), & (x^2 + y^2 + z^2 = 1). \end{aligned}$$

Useful formulas.

- The fundamental solution of **Laplace's equation** on \mathbb{R}^2 is $G_0(x) = -\frac{1}{2\pi} \log |x|$.
- The fundamental solution of the **heat equation** $u_t = \Delta u$ in \mathbb{R}^n is $\Phi(x, t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$.
- **Kirchhoff's formula:** The solution of the three-dimensional wave equation $u_{tt} = \Delta u$ with initial values (ϕ, ψ) is given by

$$u(x_0, t_0) = \frac{\partial}{\partial t_0} \left[\frac{1}{4\pi t_0} \int_{|x-x_0|=t_0} \phi(x) dS(x) \right] + \frac{1}{4\pi t_0} \int_{|x-x_0|=t_0} \psi(x) dS(x).$$

- The inversion at the unit sphere in \mathbb{R}^n , given by $x \mapsto \bar{x} = \frac{x}{|x|^2}$, satisfies $|\bar{x} - \bar{y}| = \frac{|x-y|}{|x||y|}$.