CLASSIFICATION OF GROUPS OF ORDER 60

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Remark: This is a long problem, and there are many ways to attack various of the steps. I am not claiming this is the best way to proceed, nor the fastest. If you have suggestions or questions, please let me know.

We want to classify groups of order 60 up to isomorphism. I will assume that you are familiar with the classifications of groups of orders 12, 15, and 20.

Let G be a group of order 60. Let P be a Sylow 5-subgroup, Q be a Sylow 3-subgroup, and R be a Sylow 2-subgroup of G. We know that $P \cong Z_5$, $Q \cong Z_3$, and $R \cong Z_4$ or $Z_2 \times Z_2$. A direct application of the Sylow theorems tells us that the possibilities for the number of 5-Sylows, 3-Sylows, and 2-Sylows are:

$$n_5 = 1, 6; \qquad n_3 = 1, 4, 10; \qquad n_2 = 1, 3, 5, 15.$$

We will complete this classification by proving:

1. If $n_5 = 6$, then $G \cong A_5$.
2. If $n_5 = 1$, then $n_3 = 1$ or 4.
3. If $n_5 = 1$ and $n_3 = 4$, then $G \cong A_4 \times Z_5$.
4. If $n_5 = 1$ and $n_3 = 1$, then $G \cong Z_{15} \rtimes Z_4$ or $G \cong Z_{15} \rtimes (Z_2 \times Z_2)$ for some semidirect product.
5. There are exactly six non-isomorphic semidirect products $Z_{15} \rtimes Z_4$.
6. There are exactly five non-isomorphic semidirect products $Z_{15} \rtimes (Z_2 \times Z_2)$.

This will complete the list of the 13 isomorphism classes of groups of order 60.

1 Case $n_5 = 6$

There are various ways to conclude that in this case $G \cong A_5$, but they all involve a fair amount of counting and case checking. The textbook first proves that any group of order 60 with more than one 5-Sylow is simple (Proposition 21 on page 145) and then that any simple group of order 60 is isomorphic to A_5 (Proposition 23 on page 145). We will take a different approach.

Recall the following

Lemma 1.1. *Let G be a group and $H \leq G$ with $|G : H| = m$. Let N be the largest normal subgroup of G contained in H. Then there is a group homomorphism $\phi : G \to S_m$ with kernel N. In particular G/N is isomorphic to a subgroup of S_m.*

For our problem, first we will prove that there is a subgroup $H \leq G$ of order 12 (and index 5). Then we will prove that the largest normal subgroup of G contained in H is $\{1\}$. From there we will conclude that $G \cong A_5$.

1.1 Proof that there exists a subgroup H with order 12

- Since $n_5 = 6$, there are 24 elements of order 5.

- If $n_3 = 1$ or 4, then $N_G(Q) = 15$ or 60. In particular, Q is normalized by a 5-Sylow, so we can build a subgroup of order 15. But Z_{15} is the only group of order 15, which has a normal 5-Sylow. Hence $|N_G(P)| \geq 15$ and $n_5 \leq 4$. Contradiction. We now know that $n_3 = 10$ and there are 20 elements of order 3.

- If $n_2 = 1$ or 3, we can do a very similar argument using 2-Sylows instead of 3-Sylows and we also reach a contradiction. We now know that $n_2 = 5$ or 15.

- Let R_1 and R_2 be two different 2-Sylows, let $A = R_1 \cap R_2$ and assume that $|A| = 2$. Let $B = N_G(A)$. Since $R_1 \leq B$ we know that 4 divides $|B|$. Moreover $R_1 \cup R_2 \subseteq B$, so we know that $|B| > 4$. Hence $|B| = 12, 20$ or 60.

  - If $|B| = 12$, then we found our subgroup of order 12.
  - If $|B| = 20$, then a 5-Sylow of B is normal in B. (We know this happens in every group of order 20 because we have classified them). But this implies that $|N_G(P)| \geq 20$ and $n_2 \leq 3$. Contradiction.
  - If $|B| = 60$, then we have found an element $a \in G$ with order 2 and such that $< a > \trianglelefteq G$. We know this implies that $a \in Z(G)$. It is easy to check that for every $x \in G$ with order 3, ax has order 6; and for every $x \in G$ with order 5, ax has order 10. This produces 20 elements of order 6 and 24 elements of order 10, getting a contradiction.

  In short, we have either found a subgroup of order 12 or reached a contradiction.

- If we do not have our subgroup of order 12 yet, we may assume that any pair of distinct 2-Sylows interesect trivially, and if $n_2 = 15$ they produce 45 distinct elements of order 2 or 4, getting another contradiction. Therefore $n_2 = 5$ and $|N_G(R)| = 12$. We found a subgroup of order 12.

1.2 Calculation of N for a subgroup H with order 12

Let N be the largest normal subgroup of G contained in H. Since $|H| = 12$, we know that $|N| = 1, 2, 3, 4, 6$ or 12.

- If $|N| = 3, 6$ or 12, then N must contain a Sylow 3-subgroup of G. Since N is normal and all 3-Sylows are conjugate, it must contain all of them. But there are 20 elements of order 3. Contradiction.

- If $|N| = 4$ then N would be a normal 2–Sylow, and we would have $n_2 = 1$. Contradiction.

- If $|N| = 2$ then $N \leq Z(G)$ and we get a contradiction as we did a few paragraphs above.

Hence $|N| = 1$.

1.3 Last step

By applying Lemma 1.1 to the subgroup H we found with order 12, we conclude that G is isomorphic to a subgroup (of order 60) of S_5. But we know that A_5 is the only subgroup of S_5 with index 2 (cfr. a homework problem). Hence $G \cong A_5$.

2 If $n_5 = 1$, then $n_3 \neq 10$

Since $n_5 = 1$, P is normal. Hence PQ is a subgroup of G with order 15. The only group of order 15 is Z_{15}, which has a normal 3-Sylow. Hence Q is normal in PQ, $|N_G(Q)| \geq 15$ and $n_3 \leq 4$.

3 Case $n_5 = 1$ and $n_3 = 4$

We will first prove that there is a subgroup of G isomorphic to A_4. Then we will prove that it is normal. Finally we will conclude that $G \cong Z_5 \times A_4$.

3.1 Proof that there is a subgroup H isomorphic to A_4.

- We have assumed there are four 3-Sylows Q_1, Q_2, Q_3, and Q_4. Since P is normal, for each one of them we can construct the group PQ_i, which is of order 15, and hence isomorphic to Z_{15}. If $i \neq j$, then $PQ_i \cap PQ_j = P$, so each PQ_i produces 10 distinct elements not in P, which have orders 3 or 15, for a total of 40 elements with order 3 or 15.

- Since P is normal, we can construct the subgroup PR, which has order 20. There are 20 elements with orders not 3 or 15, so at most 20 elements with an order which divides 20, so at most one subgroup of order 20. Hence PR is the unique subgroup of order 20, hence characteristic, and hence normal. Since all 2-Sylows are conjugate, they are all subgroups of PR. But the number of 2-Sylows of a group of order 20 is 1 or 5, hence $n_2 = 1$ or 5. Therefore

$|G : N_G(R)| = 1$ or 5, and $|N_G(R)| = 12$ or 60 In particular R is normalized by a Sylow 3-subgroup of G. This is true for any other 2–Sylow as well.

- Assume that Q_i normalizes R. Then we can build the group RQ_i, which has order 12. Since $|G : N_G(Q_i)| = 4$, $|N_G(Q_i)| = 15$, and Q_i cannot be normal in RQ_i. We know that there is a unique group of order 12 without a normal 3-Sylow, namely A_4. Hence we have shown that $RQ_i \cong A_4$. Call $H = RQ_i$.

3.2 Proof that H is normal.

- Since $n_3(A_4) = 4$, all four 3–Sylows of G have to be contained in RQ_i, and in particular $H = RQ_j$ for any j.
- The above steps will also be true for any other 2–Sylow subgroup of G. Let R' be a different 2–Sylow. Then we know that $H' = R'Q_1$ is also a subgroup isomorphic to A_4. Now notice that both H and H' have eight elements of order 3. There are only 8 elements of order 3 in G in total, so $|H \cap H'| \geq 8$, and due to their orders being 12, we conclude that $H = H'$.
- Finally, notice that any subgroup of order 12 will be generated by some 2-Sylow and some 3-Sylow. Hence H is the only subgroup of order 12, hence characteristic, and hence normal.

3.3 Last step

We know that both H and P are normal. Also, $H \cong A_4$ and $P \cong Z_5$. Looking at their orders, $H \cap P = \{1\}$ and $HP = G$. We conclude $G = A_4 \times Z_5$.

4 Case $n_5 = 1$ and $n_3 = 1$

In this case $P \trianglelefteq G$ and $Q \trianglelefteq G$ so that $PQ \trianglelefteq G$. Moreover $PQ \cong P \times Q \cong Z_5 \times Z_3 \cong Z_{15}$.

Notice also that $PQ \cap R = \{1\}$ and that $(PQ)R = G$. We conclude that $G \cong PQ \rtimes R$, i.e. $G \cong Z_{15} \rtimes Z_4$ or $G \cong Z_{15} \rtimes (Z_2 \times Z_2)$, and we are left with classifying these semidirect products.

5 Classification of semidirect products $Z_{15} \rtimes Z_4$

We want to classify all possible groups of the form $Z_{15} \rtimes_\phi Z_4$ for different group homomorphisms $\phi : Z_4 \to \mathrm{Aut}(Z_{15})$.

5.1 Find all possible homomorphisms ϕ

Let us write $Z_4 =< x >$. We know that $Z_{15} \cong Z_5 \times Z_3$. Let us write $Z_5 =< a >$ and $Z_3 =< b >$. We also know that $\mathrm{Aut}(Z_{15}) \cong \mathrm{Aut}(Z_5) \times \mathrm{Aut}(Z_3)$, and that $\mathrm{Aut}(Z_5) \cong Z_4$ and $\mathrm{Aut}(Z_3) \cong Z_2$. By direct search we find generators $\mathrm{Aut}(Z_5) =< \sigma >$ and $\mathrm{Aut}(Z_3) =< \tau >$ defined by $\sigma(a) = a^2$ and $\tau(b) = b^{-1}$.

A group homomorphism $\phi : Z_4 \to \mathrm{Aut}(Z_5) \times \mathrm{Aut}(Z_3)$ will be defined by $\phi(x)$. It is easy to see that there are six possible homomorphisms:

$\phi_1(x) = (\mathrm{id}, \mathrm{id})$	$\phi_2(x) = (\sigma^2, \mathrm{id})$	$\phi_3(x) = (\sigma, \mathrm{id})$
$\phi_4(x) = (\mathrm{id}, \tau)$	$\phi_5(x) = (\sigma^2, \tau)$	$\phi_6(x) = (\sigma, \tau)$

Let us define $G_i = Z_{15} \rtimes_{\phi_i} Z_4$ for each $i = 1, \ldots, 6$. We have found six groups, but we do not know yet whether there are any isomorphic groups in this list.

We have presentations for each one of these groups:

$$G_1 =< a, b, x \mid a^5 = b^3 = x^4 = 1,\ ab = ba,\ xax^{-1} = a,\ xbx^{-1} = b >$$
$$G_2 =< a, b, x \mid a^5 = b^3 = x^4 = 1,\ ab = ba,\ xax^{-1} = a^{-1},\ xbx^{-1} = b >$$
$$G_3 =< a, b, x \mid a^5 = b^3 = x^4 = 1,\ ab = ba,\ xax^{-1} = a^2,\ xbx^{-1} = b >$$
$$G_4 =< a, b, x \mid a^5 = b^3 = x^4 = 1,\ ab = ba,\ xax^{-1} = a,\ xbx^{-1} = b^{-1} >$$
$$G_5 =< a, b, x \mid a^5 = b^3 = x^4 = 1,\ ab = ba,\ xax^{-1} = a^{-1},\ xbx^{-1} = b^{-1} >$$
$$G_6 =< a, b, x \mid a^5 = b^3 = x^4 = 1,\ ab = ba,\ xax^{-1} = a^2,\ xbx^{-1} = b^{-1} >$$

5.2 Proof that the six obtained groups are not isomorphic to each other

These six groups are indeed non-isomorphic. There are various ways to see this. For every i, let us write $\phi_i(x) = (\alpha_i(x), \beta_i(x)) \in \mathrm{Aut}(Z_5) \times \mathrm{Aut}(Z_3)$. Then a direct calculation shows that

$$C_{G_i}(Z_5) \cap Z_4 = \ker \alpha_i$$
$$C_{G_i}(Z_3) \cap Z_4 = \ker \beta_i$$

In words, $\ker \alpha_i$ is the intersection of the centralizer of the *unique* 5-Sylow with a 2-Sylow (and similarly for $\ker \beta_i$). Now let's assume that $F : G_i \to G_j$ is an isomorphism. Let P be the 5-Sylow of G_i and let P' be the 5-Sylow of G_j. Since they are characteristic, $F(P) = P'$, hence $F(C_{G_i}(P)) = C_{G_j}(P')$. Now let R be some 2-Sylow of G_i and let R' be some 2-Sylow of G_j. We know that $F(R)$ and R' are conjugate in G_j, i.e. there is an inner automorphism of G_j, call it H, such that $H(F(R)) = R'$. Finally, notice that

$$C_{G_i}(Q) \cap R \cong HF(C_{G_i}(Q) \cap R) = HF(C_{G_i}(Q)) \cap HF(R) = C_{G_j}(Q') \cap R'.$$

As a consequence, if $\ker \alpha_i \not\cong \ker \alpha_j$, then $G_i \not\cong G_j$. Similarly if $\ker \beta_i \not\cong \ker \beta_j$, then $G_i \not\cong G_j$. This is enough to conclude that these six groups are all distinct.

5.3 Identification of these groups

First notice that $G_1 \cong Z_{60}$. Also, $G_3 \cong \mathrm{Hol}(Z_5) \times Z_3$. (For the definition of the holomorph of a group, see problem 5 in page 179.)

To describe with few words a few of the other, let us introduce some notation. Let H be any abelian group. Let $K = Z_{2m} =< x >$ for some integer m. Then we define the group $H \odot K := H \rtimes_\phi K$ to be the semidirect product of H and K via the homomorphism ϕ defined by $\phi_x(a) = a^{-1}$ for all $a \in H$. (Notice that this is always well defined under these hypothesis.) With this notation we have

$$G_2 \cong (Z_5 \odot Z_4) \times Z_3$$
$$G_4 \cong Z_5 \times (Z_3 \odot Z_4)$$
$$G_5 \cong Z_{15} \odot Z_4$$

As for G_6 we do not have a short name. An alternative presentation for it is

$$G_6 =< c, x \mid c^{15} = x^4 = 1,\ xcx^{-1} = c^2 > .$$

6 Classification of semidirect products $Z_{15} \rtimes (Z_2 \times Z_2)$

To classify these groups, we need to study all possible group homomorphisms

$$\phi : Z_2 \times Z_2 \to \mathrm{Aut}(Z_{15}) \cong \mathrm{Aut}(Z_5) \times \mathrm{Aut}(Z_3)$$

Let us write $Z_2 \times Z_2 =< x > \times < y >$. We will also use the same notation for Z_5 and Z_3 as in the previous section.

There are 16 such homomorphisms ϕ but plenty of them produce isomorphic semidirect products. Let us recall the following result (from problem 7 in the handout):

Lemma 6.1. *Let H and K be finite groups. Assume that H is abelian and that $(|H|, |K|) = 1$. Let $\phi_1, \phi_2 : K \to \mathrm{Aut}(H)$ be two group homomorphisms. If $\ker \phi_1 \ncong \ker \phi_2$, then $H \rtimes_{\phi_1} K \ncong H \rtimes_{\phi_2} K$.*

There are three options for $\ker \phi$.

6.1 Case $|\ker \phi| = 4$

In this case ϕ is trivial and the semidirect product is a direct product. This produces a new group:

$$G_7 = Z_{15} \times (Z_2 \times Z_2) \cong Z_{30} \times Z_2.$$

6.2 Case $|\ker \phi| = 2$

There are 3 options for the kernel of ϕ, but they are the same up to a change on the name of the generators, and they will produce isomorphic semidirect products, so we may assume $\ker \phi =< x >$. (For a formal justification, see problem 7 in the handout.) For any such ϕ we notice that

$$Z_{15} \rtimes (Z_2 \times Z_2) = Z_{15} \rtimes (< x > \times < y >) = (Z_{15} \rtimes < y >) \times < x >$$

for some semidirect product. In other words, our group is a direct product of a group of order 2 and a non-abelian group of order 30. Since we have classified all groups of order 30, we know this gives us three possibilities:

$$\begin{aligned} G_8 &= (D_{10} \times Z_3) \times Z_2 \cong D_{10} \times Z_6 \\ G_9 &= (Z_5 \times D_6) \times Z_2 \cong Z_{10} \times D_6 \\ G_{10} &= D_{30} \times Z_2 \cong D_{60} \end{aligned}$$

To check that no pair of these three groups are isomorphic to each other, calculate the centralizer of their 5-Sylows and of their 3-Sylows. (Since the 5-Sylow and the 3-Sylow are unique, we can distinguish them by their centralizers.)

6.3 Case $|\ker \phi| = 1$

In this case $Z_2 \times Z_2 \cong \operatorname{Image} \phi \leq \operatorname{Aut} Z_5 \times \operatorname{Aut} Z_3$. There is a unique subgroup of $\operatorname{Aut} Z_5 \times \operatorname{Aut} Z_3$ isomorphic to $Z_2 \times Z_2$, namely $< \sigma^2, \tau >$. There are 6 possible homomorphisms ϕ, but they correspond to renaming the generators of $Z_2 \times Z_2$ and will all produce isomorphic semidirect products. (Again, for a formal justification, see problem 7 in the handout.) Hence we get one new group. One possible presentation is

$$G_{11} =< a, b, x, y \mid a^5 = b^3 = x^2 = y^2,\ ab = ba, xy = yx,\ xax^{-1} = a^{-1}, xbx^{-1} = b,\ yay^{-1} = a, yby^{-1} = b^{-1} >$$

We also notice that $G_{11} =< a, x > \times < b, y > \cong D_{10} \times D_6$.

7 Summary

There are 13 groups of order 60 up to isomorphism:

$$
\begin{aligned}
G_1 &= Z_{60} \\
G_2 &= (Z_5 \odot Z_4) \times Z_3 \\
G_3 &= \mathrm{Hol}(Z_5) \times Z_3 \\
G_4 &= Z_5 \times (Z_3 \odot Z_4) \\
G_5 &= Z_{15} \odot Z_4 \\
G_6 &= < c, x \mid c^{15} = x^4 = 1,\ xcx^{-1} = c^2 > \\
G_7 &= Z_{30} \times Z_2 \\
G_8 &= D_{10} \times Z_6 \\
G_9 &= Z_{10} \times D_6 \\
G_{10} &= D_{60} \\
G_{11} &= D_{10} \times D_6 \\
G_{12} &= A_5 \\
G_{13} &= A_4 \times Z_5
\end{aligned}
$$