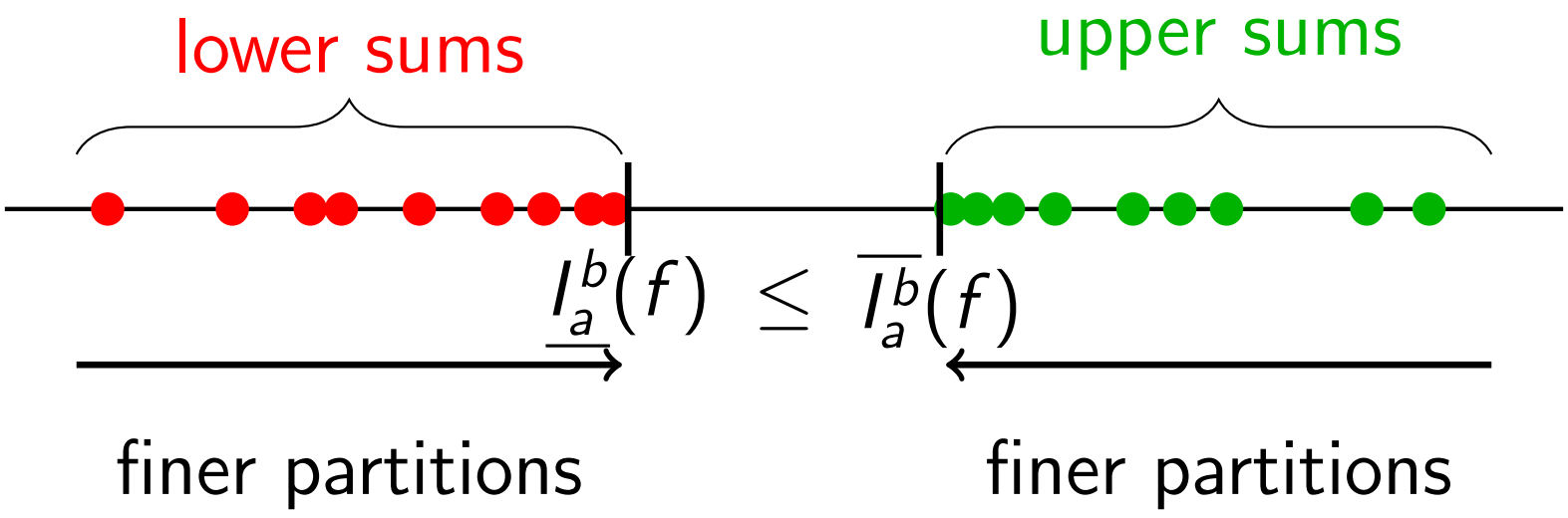
- Today: Integrals as limit
- FRIDAY: Antiderivatives and functions defined as integrals **(Videos 8.1, 8.2)**



An alternative definition

Recall

Let $A \subseteq \mathbb{R}$. The supremum of A as the only real number S such that...

- S is an upper bound of A.
- $\forall \varepsilon > 0, \quad \exists x \in A \quad$ such that $\quad S - \varepsilon < x$.

Complete the following alternative definition of lower integral:

Let f be a bounded function on the interval $[a, b]$.
$\underline{I}_a^b(f)$ is the only real number that satisfies these two properties:

1. $\forall$ partition P of $[a, b]$, ...
2. $\forall \varepsilon > 0$, ...

Do the same thing for upper integral.

The norm of a partition

1. Construct a partition P of $[0, 1]$ such that $||P|| = \frac{\pi}{10}$.
2. Construct a sequence of partitions of $[0, 1]$

$$P_1, P_2, P_3, \ldots$$

as simple as possible, such that $\lim_{n\to\infty} ||P_n|| = 0$.

3. Construct a *different* sequence of partitions of $[0, 1]$

$$Q_1, Q_2, Q_3, \ldots$$

such that $\lim_{n\to\infty} ||Q_n|| = 0$.

Compute $\int_1^2 x^2 dx$ using Riemann sums

Let $f(x) = x^2$ on $[1, 2]$. Let P_n be the partition that breaks $[1, 2]$ into n subintervals of equal length.

1. Write a explicit formula for P_n.
2. What is Δx_i?
3. Write the Riemann sum $S^*_{P_n}(f)$ with sigma notation (choose x_i^* as the right endpoint).
4. Add the sum
5. Compute $\lim_{n\to\infty} S^*_{P_n}(f)$.
6. Repeat the last 3 questions when we choose x_i^* as the left endpoint.

Helpful identities: $\sum_{i=1}^{N} i = \frac{N(N+1)}{2}$, $\sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6}$

Example 2

Consider the function f defined on $[0, 1]$:

$$f(x) = \begin{cases} 1/2 & \text{if } 0 \leq x < 1/2 \\ 1 & \text{if } 1/2 \leq x \leq 1 \text{ and } x \in \mathbb{Q} \\ 0 & \text{if } 1/2 \leq x \leq 1 \text{ and } x \notin \mathbb{Q} \end{cases}$$

1. Draw a picture!
2. Let $P = \{0, 0.2, 0.4, 0.6, 0.8, 1\}$. Calculate $L_P(f)$ and $U_P(f)$.
3. Construct a partition P s.t. $L_P(f) = .24$ and $U_P(f) = .76$
4. What is the upper integral, $\overline{I_0^1}(f)$?
5. What is the lower integral, $\underline{I_0^1}(f)$?
6. Is f integrable on $[0, 1]$?