Unique continuation for the vacuum Einstein equations.

Spyros Alexakis*

Abstract

We derive a unique continuation theorem for the vacuum Einstein equations. Our method of proof utilizes Carleman estimates (most importantly one obtained recently by Ionescu and Klainerman), but also relies strongly on certain geometric gauge constructions which make it possible to address this problem via such estimates. We indicate how our method can be used more broadly to derive unique continuation for Einstein's equations from Carleman estimates for the wave operator.

1 Introduction.

The main result of this paper is a unique continuation theorem for the vacuum Einstein equations across bifurcate horizons. For the reader's convenience we start with a rough description of the main theorem: Let $(M, \mathbf{g}), (\tilde{M}, \tilde{\mathbf{g}})$ be two vacuum space-times (meaning that $Ric(\mathbf{g}) = 0, Ric(\tilde{\mathbf{g}}) = 0$), both being the global hyperbolic developments of space-like hypersurfaces $\Sigma, \tilde{\Sigma}$ respectively. Let $B, \tilde{B}$ be topological open balls in $\Sigma, \tilde{\Sigma}$ and denote by $F, \tilde{F}$ the sets of points in $M, \tilde{M}$ respectively that can be joined to $B, \tilde{B}$ by time-like curves (future-directed or past-directed). The regions $F, \tilde{F}$ have boundaries which we denote by $\mathcal{H}, \tilde{\mathcal{H}}$. One can visualize $\mathcal{H}$ (resp. $\tilde{\mathcal{H}}$) as the consisting of the (non-smooth) union of two truncated null cones: One truncated cone emanates from ∂B (resp. $\partial\tilde{B}$) towards the future and the other truncated cone emanates from ∂B (resp. $\partial\tilde{B}$) towards the past. The region F (resp. $\tilde{F}$) can be thought of as the "inside" of $\mathcal{H}$ (resp. $\tilde{\mathcal{H}}$); the region $M \setminus F$ (resp. $\tilde{M} \setminus \tilde{F}$) can be thought of as the "outside" of $\mathcal{H}$ (resp. $\tilde{\mathcal{H}}$). In simple language, we then prove that if the "insides" $(F, \mathbf{g}), (\tilde{F}, \tilde{\mathbf{g}})$ are isometric then two open subsets of the "outsides" $(M \setminus F, \mathbf{g}), (\tilde{M} \setminus \tilde{F}, \tilde{\mathbf{g}})$ must also be isometric.

Unique continuation problems for PDEs have a long history, see [5] for a general discussion. However, such results for geometric equations (typically equations in the curvature) have only received attention recently (see [1] where Biquard derived unique continuation results for Einstein metrics of *Riemannian* signature). Such theorems are often proven using Carleman-type estimates and indeed we will follow this approach in the present paper. The relevant Carleman estimate that we use comes from a recent paper of Ionescu and Klainerman [3].

In the rest of this introduction we state Theorem 1.1 in detail, and make some remarks on certain extensions of this result that can be derived by a straightforward modification of the proof (see Theorem 1.2). We then briefly outline some of the arguments in the proof of Theorem 1.1. In section 2 we prove Theorem 1.1. For completeness, in section 3 we present a derivation of the result of Theorem 1.1 under a minimal set of hypotheses. It is worth noting the analogy of the calculations in section 3 with the ones of Rendall in [4].

The main result: Our theorem deals with vacuum space-times which are maximal developments of incomplete initial data sets: We will be interested in $\mathcal{C}^4$ space-times $(M, \mathbf{g})$ which admit a Caucy hypersurface $\Sigma_0 \subset M$ where $(\Sigma_0, \mathbf{g})$ is a $\mathcal{C}^4$-Riemannian manifold with boundary, $\partial\Sigma_0 = S$ is topologically a 2-sphere and $\mathbf{g}$

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extends in a $\mathcal{C}^4$-fashion to $\partial\Sigma_0$. It follows that in a small relatively open neighborhood of S, M will have a boundary $\mathcal{H}$ consisting of the union of two null hypersurfaces $\mathcal{H}^+$ and $\mathcal{H}^-$ each of which is ruled by null geodesic rays, so that $\mathcal{H}^+$ and $\mathcal{H}^-$ intersect transversely at S. These two future and past horizons $\mathcal{H}^+, \mathcal{H}^-$ are thus each diffeomorphic to $\mathbb{S}^2 \times [0, \infty)$ and the metric $\mathbf{g}$ restricted to $\mathcal{H}^+, \mathcal{H}^-$ is degenerate. Following [3] we call S the *bifurcate sphere* and the union $\mathcal{H}^+ \bigcup \mathcal{H}^-$ the *bifurcate horizon.*

Our main theorem is then the following:

Theorem 1.1. *Let $(M, \mathbf{g})$, $(\tilde{M}, \tilde{\mathbf{g}})$ be two vacuum space-times ($Ric(\mathbf{g}) = 0$ and $Ric(\tilde{\mathbf{g}}) = 0$) as described above. Denote by $S, \tilde{S}$ their bifurcate spheres and by $\mathcal{H}^+ \bigcup \mathcal{H}^-$, $\tilde{\mathcal{H}}^+ \bigcup \tilde{\mathcal{H}}^-$ their bifurcate horizons.*

Assume that there exist points $P \in S, \tilde{P} \in \tilde{S}$ and relatively open sets $\Omega \subset M, \tilde{\Omega} \subset \tilde{M}$ with $P \in \Omega$, $\tilde{P} \in \tilde{\Omega}$ containing $S, \tilde{S}$, and a diffeomorphism $\Phi : \Omega \to \tilde{\Omega}$ so that $\mathbf{g} - \Phi^*\tilde{\mathbf{g}}$ vanishes to third order on $(\mathcal{H}^+ \bigcup \mathcal{H}^-) \bigcap \Omega$.[1] *Then the metrics $\mathbf{g}, \tilde{\mathbf{g}}$ are isometric in some relatively open neighborhoods of $P, \tilde{P}$ in $M, \tilde{M}$.*

Remark 1: It turns out that the above result can be derived under substantially weaker hypotheses on the diffeomorphism Φ: We will show in section 3 that it suffices to only assume that the induced conformal structures of the horizons $\mathcal{H}, \tilde{\mathcal{H}}$ agree near $P, \tilde{P}$, along with certain requirements on the metrics $\mathbf{g}, \tilde{\mathbf{g}}$ restricted to the spheres $S \bigcap \Omega, \tilde{S} \bigcap \tilde{\Omega}$. This will essentially follow by a careful analysis of the Einstein equations on characteristic hypersurfaces, and is in complete analogy with [4].

Remark 2: We note that our methods can actually show the following extension of the above: Assume that $\mathcal{H}^+, \mathcal{H}^-$ are future/past-complete and also that the metric $\mathbf{g}$ satisfies certain $\mathcal{C}^1$ bounds near the horizon $\mathcal{H}^+ \bigcup \mathcal{H}^-$ (in the interest of brevity we will not make this statement more precise); then if there exists a diffeomorphism $\Phi : M \to \tilde{M}$ for which $\mathbf{g} - \Phi^*\tilde{\mathbf{g}}$ vanishes to third order on the entire horizon $\mathcal{H}^+ \bigcup \mathcal{H}^-$, the space-times $(M, \mathbf{g})$, $(\tilde{M}, \tilde{\mathbf{g}})$ will be isometric in open neighborhoods of the horizons $\mathcal{H}^+ \bigcup \mathcal{H}^-$. We will make a remark further down to point out why this is true.

Remark 3: In fact, the method we introduce to show Theorem 1.1 can be applied more widely to show unique continuation across other types of hypersurfaces; we indicate how it can be readily adapted to prove unique continuation for the vacuum Einstein equations across any smooth time-like hypersurface $\mathcal{H}$, provided Hörmander's *strong pseudo-convexity condition* holds for $\mathcal{H}$: The notion of strong pseudo-convexity is defined for very general classes of operators (see the discussion in [5]), but for simplicity we will explain it only for the wave operator across a smooth time-like surface $\mathcal{H}$: Consider a Lorentzian manifold $(M, \mathbf{g})$ (with an associated wave operator $\Box_g$) and a smooth time-like hypersurface $\mathcal{H} \subset M$ which divides M into regions M^+, M^-. We then say that M^- is strongly pseudo-convex with respect to the wave operator $\Box_g$ (or equivalently with respect to the metric g) near $P \in \mathcal{H}$ if there exists an open neighborhood Ω of P so that every *null geodesic* in Ω which is tangent to $\mathcal{H}$ at some point $P' \in \Omega \bigcap \mathcal{H}$ lies entirely in M^+, and it only touches $\mathcal{H}$ at P', with first order of contact.

The next theorem can be proven by a straightforward adaptation of the method of proof of Theorem 1.1:

Theorem 1.2. *Let $\mathbf{g}, \tilde{\mathbf{g}}$ be two $\mathcal{C}^4$ Lorentzian metrics defined over a domain $\Omega \subset \mathbb{R}^4$ satisfying the vacuum Einstein equations: $Ric(\mathbf{g}) = 0, Ric(\tilde{\mathbf{g}}) = 0$. Let $\mathcal{H}$ be a smooth time-like hypersurface which divides Ω into two subdomains Ω_1, Ω_2, and assume that $\mathbf{g} = \tilde{\mathbf{g}}$ in Ω_2; assume also that Ω_1 satisfies the strong pseudo-convexity condition with respect to the metric $\mathbf{g}$ at $P \in \mathcal{H}$. Then $\mathbf{g}, \tilde{\mathbf{g}}$ are isometric in some relatively open neighborhoods of P into Ω_1.*

Remark 4: In the proof of Theorem 1.1 we introduce a general method which uses Carleman Estimates for the wave operator to derive unique continuation for solutions

[1]By this we mean that the tensor $\mathbf{g} - \Phi^*\tilde{\mathbf{g}}$ and all its first and second derivatives vanish on $\mathcal{H}^+ \bigcup \mathcal{H}^-$.

of the vacuum Einstein equations. Thus, Theorem 1.2 essentially follows by applying this technique to the classical Carleman estimate of Hörmander (see Theorem 4 in [5], or section 28 in [2] for more details). We will highlight (using separate remarks) along the course of the proof of Theorem 1.1 the instances where the arguments must be slightly altered in order to derive Theorem 1.2.

Discussion of the Proof of Theorem 1.1: There are several interesting aspects of applying a Carleman-type estimate for wave operators to solutions of Einstein's equations in vacuum; in particular, the geometric nature of the equations comes starkly into play.

Clearly, in order to reduce the problem to applying a Carleman estimate we must fix a "canonical" gauge in which to express the two metrics $\mathbf{g}, \tilde{\mathbf{g}}$ and then to use the Einstein equations to derive a PDE on the *difference* of the two metrics. The problem is then reduced to showing that this difference must vanish in an open neighborhood of the bifurcate sphere by applying the Carleman estimate to this PDE.

Now, the Ricci curvature is a (non-linear) second order partial differential operator acting on the metric; *in wave coordinates* the Ricci curvature has the wave operator as its principal symbol. One would therefore ideally wish to fix the gauge by picking wave coordinates for $\mathbf{g}, \tilde{\mathbf{g}}$ and then subtracting the corresponding equations $Ric(\mathbf{g}) = 0, Ric(\tilde{\mathbf{g}}) = 0$. However this is *not possible*: Finding wave coordinates in this setting is equivalent to solving a hyperbolic PDE which is *ill-posed* (in the sense that it does not have a solution in general). Therefore a different choice of gauge must be made, and also a way of circumventing the fact that the principle symbol of the Ricci operator will *not* be the wave operator must be found.

Our remedy to these problems is to introduce *double Fermi coordinates* and to work with a wave equation for the curvature tensors $\mathbf{R}, \tilde{\mathbf{R}}$ of the metrics $\mathbf{g}, \tilde{\mathbf{g}}$: $\Box_{\mathbf{g}}\mathbf{R} = \mathbf{R} * \mathbf{R}$, $\Box_{\tilde{\mathbf{g}}}\tilde{\mathbf{R}} = \tilde{\mathbf{R}} * \tilde{\mathbf{R}}$.[2] Double Fermi coordinates are constructed by considering a particular null vector field V on $\mathcal{H}^+$ (obtained through parallel transport along the null generators of $\mathcal{H}^+$) that points into M, and then constructing the (arc-length parametrized) null geodesics that emanate from this vector field V. This choice of gauge induces a canonical diffeomorphism Ψ between the space-times $(M, \mathbf{g}), (\tilde{M}, \tilde{\mathbf{g}})$ (locally near $P, \tilde{P}$) which reduces the problem to comparing the metrics $\mathbf{g}, \Psi^*\tilde{\mathbf{g}}$ over M. We then subtract the two wave equations above and derive a wave equation, (2.7), for the *difference* $\mathbf{T_{abcd}}$ of the curvature tensors $\mathbf{R_{abcd}}, \tilde{\mathbf{R}}_{\mathbf{abcd}}$ of the metrics $\mathbf{g}, \Psi^*\tilde{\mathbf{g}}$. However this equation (2.7) also includes terms involving the difference $\mathbf{d_{ab}}$ of the two metrics $\mathbf{g}, \Psi^*\tilde{\mathbf{g}}$, the difference $\mathbf{G_{ab,c}}$ of their connection coefficients,[3] and also the derivatives of $\mathbf{G_{ab,c}}$.

The problem then reduces to controlling the weighted L^2-norms of these extra terms by the weighted L^2-norms of the terms $\mathbf{T}, \partial\mathbf{T}$ and $\Box\mathbf{T}$. Now, *in the double Fermi coordinates* we have constructed, the metric is related to the curvature via an *ODE*, (2.1). This allows us to control the weighted L^2-norms of $\mathbf{d}$ and $\mathbf{G}$ in (2.13) by weighted L^2-norms of $\mathbf{T}$ and $\partial\mathbf{T}$, which can then be absorbed into the Carleman inequality (2.12). However, the equation (2.7) also contains certain "bad terms" involving *derivatives* of $\mathbf{G_{ab,c}}$; in this setting a straightforward application of the ODE relation would not allow us to control the norm of these terms by the norms of the terms $\mathbf{T}, \partial\mathbf{T}$ and $\Box\mathbf{T}$. At this point we make use of the precise algebraic form of the "bad terms" (two indices are traced) and another special property of our coordinate system; in particular *in double Fermi coordinates*, the "bad terms" involve no second derivatives of $\mathbf{d}$ in certain "bad directions". This fact, coupled with standard elliptic estimates on the level sets of the Carleman weight function f_ϵ and the algebraic identities of the curvature tensor, allow us to controll the weighted L^2-norm of the "bad terms" by quantities which are allowed in our Carleman estimate. This enables us to close up the argument and derive that $\mathbf{T} = 0$ (and then $\mathbf{d} = 0$) from our Carleman

[2] These equations follow from the equation $Ric = 0$ via the Bianchi identities.

[3] I.e. the difference of their Christoffel symbols $\Gamma_{ab,c}, \tilde{\Gamma}_{ab,c}$.

inequality by a standard argument (see [2] or [5]).

We now introduce some notational conventions.

Conventions: We wish to introduce a dichotomy between smooth tensor fields defined over $M, \tilde{M}$ and the components of these smooth tensor fields. We will denote abstract tensor fields with bold letters e. g. $\mathbf{A}, \mathbf{B}$ or if we wish to designate their type or the position of their indices we will also include the indices: e. g. $\mathbf{A}_\alpha^\beta$ is a $(1,1)$-tensor field and $\mathbf{B}_{\alpha\beta}{}^\gamma$ is a $(1,2)$-tensor field. On the other hand, once we have constructed a frame field (say X^0, X^1, X^2, X^3) for our manifold below we will denote by A_a^b or $B_{ab}{}^c$ the components of the above tensor fields with respect to this frame. For example, g_{12} will stand for the component of the metric tensor (which is a $(0,2)$-tensor–with lower indices) evaluated for the vectors X^1, X^2; in other words $g_{12} = g(X^1, X^2)$.[4] Furthermore, throughout the next section, we will have many generic tensor fields appearing as coefficients in equations below (e.g. the term $L'^{yu}_{ab} d_{yu}$ in (2.3)); unless stated otherwise, these will be $\mathcal{C}^2$-tensor fields over Ω. Also, unless we explicitly write out a different summation of indices, repeated upper and lower indices (as in L'^{yu}_{ab} in (2.3)) will mean that we apply the Einstein summation convention and sum over all values $^1, ^2, ^3, ^4$, $_1, _2, _3, _4$ that we can give those indices. Finally, we introduce the convention that in section 2 all estimates involving the parameter λ will hold for λ large enough, and all constants appearing in the estimates will be independent of λ.

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2 The proof of Theorem 1.1.

2.1 Double Fermi coordinates and a PDE-ODE system.

We will explicitly construct the desired (local, near $P, \tilde{P}$) isometry Ψ between $(M, \mathbf{g})$ and $(\tilde{M}, \tilde{\mathbf{g}})$. In order to do this, we firstly construct a useful set of coordinates outside the bifurcate horizon $\mathcal{H}^+ \bigcup \mathcal{H}^-$.

Consider the sphere S in $(M, \mathbf{g})$ and let Ω be a small relatively open neighborhood of $P \in S$; pick a pair of null vector fields U, V on S with the following two properties: Firstly, U is future-directed and tangent to $\mathcal{H}^+$ and V is past-directed and tangent to $\mathcal{H}^-$. Secondly, $g(U, V) = 1$ on all of $S \bigcap \Omega$. Now, consider the affine-parametrized null geodesics emanating from U (these will correspond to the null generators of $\mathcal{H}^+$); for each $A \in S$ we denote by l_A the null geodesic that thus emanates from P. Notice that given any coordinate system defined on $S \bigcap \Omega$, $\mathcal{Y} : S \bigcap \Omega \to \mathbb{R}^2$, we obtain a coordinate system $\mathcal{Y}' : \mathcal{H}^+ \bigcap \Omega \to \mathbb{R}^2 \times [0, 1)$.

Next, we parallel-transport the vectors V along the null geodesics l_A: Thus, for each point $Q \in \mathcal{H}^+ \bigcap \Omega$ we obtain a past-directed outward pointing null vector V_Q. Finally, consider the (affine-parametrized) null geodesics emanating from the vectors V_Q. We have thus obtained a coordinate system of the form $\mathcal{Y}'' : \Omega' \to \mathbb{R}^2 \times [0, 1) \times [0, \delta_0)$, where Ω' is some relatively open neighborhood of P in the space-time M.[5]

Definition 2.1. *For any space-time $(M, \mathbf{g})$ as in the hypothesis of Theorem 1.1 we call a system of coordinates as above "double Fermi coordinates".*

We now consider the vector fields $\tilde{U}, \tilde{V}$ in $\tilde{M}$ where $\tilde{U} = \Phi_* U, \tilde{V} = \Phi_* V$ and also the coordinate system $\tilde{\mathcal{Y}} = \mathcal{Y} \circ \Phi^{-1} : \tilde{S} \bigcap \tilde{\Omega} \to \mathbb{R}^2$. We perform the same construction as above (for δ_0 small enough) for the space-time $\tilde{M}$, obtaining a new coordinate

[4] In fact in most cases below the frame fields we will construct will be the coordinate vector fields defined by a system of coordinates.

[5] By slight abuse of notation we will denote Ω' by Ω again.

system $\tilde{\mathcal{Y}} : \tilde{\Omega} \to \mathbb{R}^2 \times [0,1) \times [0,\delta_0)$, where $\tilde{\Omega}$ is a relatively open neighborhood of $\tilde{P}$ in the space-time $\tilde{M}$. Consider the map $\Psi : \Omega \to \tilde{\Omega}$ defined by the formula:

$$\Psi = \tilde{\mathcal{Y}}^{-1} \circ \mathcal{Y}.$$

Let us pull back the metric $\tilde{\mathbf{g}}$ to M via this map: We define $\mathbf{g}' = \Psi^* \tilde{\mathbf{g}}$. We will show that $\mathbf{g}' = \mathbf{g}$ in an open neighborhood of $P \in S$. That will prove our claim.

Two remarks are in order here: Firstly, in view of the freedom of picking the vector fields U, V over S (in particular since we are only imposing the requirement $g(U,V) = 1$, we could just as well replace these vector fields by $\tau U, \frac{1}{\tau} V$) our argument below can be used to show that if there is a map $\Phi : M \longrightarrow \tilde{M}$ for which $\mathbf{g} - \Phi^* \tilde{\mathbf{g}}$ vanishes up to second order on all of $\mathcal{H}^+ \bigcup \mathcal{H}^-$, then $(M, \mathbf{g})$, $(\tilde{M}, \tilde{\mathbf{g}})$ are isometric in an open neighborhood of the whole horizon $\mathcal{H}^+ \bigcup \mathcal{H}^-$–the $\mathcal{C}^1$-bounds on $\mathbf{g}$ mentioned in Remark 2 serve to ensure that the constant ϵ below can be picked independently of τ.

Now, we wish to study the components of the metrics $\mathbf{g}, \mathbf{g}'$ with respect to the coordinate system over M that we have constructed.

Let x^1, x^2 be coordinate functions on S, defined near $P \in S$ such that $x^1(P) = x^2(P) = 0$; let x^3 be the coordinate on $\mathcal{H}^+$ defined by the null geodesics emanating from the vectors U_P through the equation $\nabla_{\frac{\partial}{\partial x^3}} \frac{\partial}{\partial x^3} = 0$. Thus we have coordinates x^1, x^2, x^3 defined on $\mathcal{H}^+$, near P. Now consider the coordinate x^0 in Ω defined by the null geodesics emanating in the direction of V_Q through the equation $\nabla_{\frac{\partial}{\partial x^0}} \frac{\partial}{\partial x^0} = 0$. Thus we obtain coordinates x^0, x^1, x^2, x^3 in the open set Ω.

Consider the metric g_{ab} (where the lower indices can take values $_{0,1,2,3}$ that correspond to the above coordinate system). Given the equation $\nabla_{\frac{\partial}{\partial x^0}} \frac{\partial}{\partial x^0} = 0$ we derive the equations: $\partial_0 g_{0a} = 0$ for $_a = {}_{0,1,2,3}$. Thus for every point in Ω we get: $g_{00} = 0, g_{01} = 0, g_{02} = 0, g_{03} = 1$. Analogously we derive that $g'_{00} = 0, g'_{01} = 0, g'_{02} = 0, g'_{03} = 1$.

Now, denote by $\Gamma_{ab,c} = \frac{1}{2}[\partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab}]$, $\Gamma'_{ab,c} = \frac{1}{2}[\partial_a g'_{bc} + \partial_b g'_{ac} - \partial_c g'_{ab}]$ the Chiristoffel symbols of the metrics g, g' in the coordinates of M that we have constructed. Consider also the curvature tensors R_{abcd}, R'_{abcd} (with 4 lower indices) of the metrics g, g' and also the Levi-Civita connections ∇, ∇' of the metrics g, g'. We define the tensors $\mathbf{d}, \mathbf{T}, \mathbf{G}, \mathbf{D}$ through the equations:[6]

$$d_{ab} = g_{ab} - g'_{ab}, T_{abcd} = R_{abcd} - R'_{abcd}, G_{ab,c} = \Gamma_{ab,c} - \Gamma'_{ab,c}, D_{ab} = \partial_s G_{ta,b} g^{st}.$$

We will prove that in some open neighborhood Ω' of S, $\mathbf{T}_{abcd} = 0$ and $\mathbf{d}_{ab} = 0$. That will prove our theorem.

Our next goal is to derive a system of equations (both PDEs and ODEs) in the tensors above.

Consider the components R_{0ab0}, R'_{0ab0} of the curvature tensors for g, g'. By the definition of curvature tensor we derive that *in the double Fermi coordinates*:

$$R_{0ab0} = \frac{1}{2} \partial^{(2)}_{00} g_{ab} + \frac{1}{4} \partial_0 g_{as} \partial_0 g_{tb} g^{st}, \tag{2.1}$$

$$R'_{0ab0} = \frac{1}{2} \partial^{(2)}_{00} g'_{ab} + \frac{1}{4} \partial_0 g'_{as} \partial_0 g'_{tb} g'^{st}. \tag{2.2}$$

Subtracting the above two equations we derive:

$$T_{0ab0} = \frac{1}{2} \partial^{(2)}_{00} d_{ab} + L^{yu}_{ab} \partial_0 d_{yu} + L'^{yu}_{ab} d_{yu}. \tag{2.3}$$

[6]Note that a tensor is specified by specifying its components relative to a frame.

Now, consider the two equations:

$$\Box_{\mathbf{g}}\mathbf{R} = \mathbf{R} * \mathbf{R}, \Box_{\mathbf{g}'}\mathbf{R}' = \mathbf{R}' * \mathbf{R}'. \tag{2.4}$$

We are going to subtract these two equations. We introduce some notation first; We let $\widetilde{\Box_g}$ be the "rough wave operator": $g^{ab}\partial_a\partial_b$ which acts on scalar-valued functions. We also note that $g^{ab} - g'^{ab} = -g^{as}d_{st}g'^{tb}$. Then, subtracting the two equations in (2.4) we derive an equation which hold for any values ${}_{a,\,b,\,c,\,d} = {}_{0,\,1,\,2,\,3}$:

$$\widetilde{\Box_g}T_{abcd} = F^{yuvx}_{abcd}T_{yuvx} + F'^{yuvxb}_{abcd}\partial_b T_{yuvx} + F''^{yu}_{abcd}d_{yu} + F'''^{yuv}_{abcd}\partial_v d_{yu} + F''''^{yu}_{abcd}D_{yu}.$$

(here the tensor fields F'', F''' are $\mathcal{C}^1, \mathcal{C}^0$ respectively).

To derive the next set of equations, we will break the tensor $\mathbf{D_{ab}}$ into the symmetric part $\mathbf{D_{(ab)}}$ ($\mathbf{D_{(ab)}} = \frac{1}{2}[\mathbf{D_{ab}} + \mathbf{D_{ba}}]$) and the antisymmetric part $\mathbf{D_{[ab]}}$ ($\mathbf{D_{[ab]}} = \frac{1}{2}[\mathbf{D_{ab}} - \mathbf{D_{ba}}]$). By the definition of the Christoffel symbols we see that $D_{(ab)}$ equals: $D_{(ab)} = -\frac{1}{2}\widetilde{\Box_g}d_{ab}$.

On the other hand we calculate: $D_{[ab]} = \frac{1}{2}[\partial^{(2)}_{sa}d_{tb} - \partial^{(2)}_{sb}d_{sa}]g^{st}$. Now, in order to derive an equation on $D_{[ab]}$ we consider the equations: $\nabla^s R_{0sab} = 0$, $\nabla'^s R'_{0sab} = 0$ and we subtract them. We derive an equation:

$$g^{st}[\partial_{s0a}d_{tb} - \partial_{s0b}d_{ta}] = C^{yu}_{ab}d_{yu} + C'^{yuv}_{ab}\partial_y d_{uv} + C''^{yuvx}_{ab}T_{yuvx} + C'''^{yu}_{ab}D_{yu} + C''''^{yuv}\partial_{0v}d_{yu}. \tag{2.5}$$

In fact, we observe that because of the form of the metric g_{ab} (with lower indices) we must have $g^{3a} = g^{a3} = 0$ for $a = 1, 2, 3$ and $g^{03} = g^{30} = 1$, therefore the above gives us an equation:

$$\partial_0 D_{[ab]} = C^{yu}_{ab}d_{yu} + C'^{yuv}_{ab}\partial_y d_{uv} + C^{yuvx}_{ab}T_{yuvx} + C'''^{yu}_{ab}D_{yu} + C''''^{yuv}_{ab}\partial_{0v}d_{yu} + \sum_{i,j=1}^{2} c^{ij}\partial_i G_{j[a,b]}. \tag{2.6}$$

Thus our system of equations is as follows:

$$\widetilde{\Box_g}T_{abcd} = F^{yuvx}_{abcd}T_{yuvx} + F^{yuvxb}_{abcd}\partial_b T_{yuvx} \tag{2.7}$$
$$+F^{yu}_{abcd}d_{yu} + F^{yuv}_{abcd}\partial_y d_{uv} + F^{yu}_{abcd}D_{yu},$$
$$T_{0ab0} = \tfrac{1}{2}\partial^{(2)}_{00}d_{ab} + L^{yu}_{ab}\partial_0 d_{yu} + L'^{yu}_{ab}d_{yu}, \tag{2.8}$$
$$G_{ab,c} = \tfrac{1}{2}[\partial_a d_{bc} + \partial_b d_{ac} - \partial_c d_{ab}], D_{ab} = g^{st}\partial_s G_{ta,b}, \tag{2.9}$$
$$D_{(ab)} = -\tfrac{1}{2}\widetilde{\Box_g}d_{ab}, \tag{2.10}$$
$$\partial_0 D_{[ab]} = C^{yu}_{ab}d_{yu} + C^{yuv}_{ab}\partial_y d_{uv} \tag{2.11}$$
$$+C^{yuvx}_{ab}T_{yuvx} + C'^{yu}_{ab}D_{yu} + C'^{yuv}_{ab}\partial^{(2)}_{0v}d_{yu} + \textstyle\sum_{i,j=1}^{2} c^{ij}\partial_i G_{j[a,b]}.$$

Remark 5: The analogue of the double Fermi coordinates in the setting of Theorem 1.2 is as follows: In this case we can pick a hypersurface $S \subset \Omega_2$ which touches $\mathcal{H}$ to first order at P, and which is still strongly pseudo-convex. We then (locally near $P \in S$) pick coordinates x^1, x^2, x^3 on S, such that x^3 is a time-like direction and x^1, x^2 are space-like. Finally, we let $\vec{\nu}$ be the (space-like) unit normal vector field to S, which points towards Ω_1 and consider the arc-length parametrized space-like geodesics that emanate from $\vec{\nu}$. For each of the metrics $\mathbf{g}, \tilde{\mathbf{g}}$, this defines a fourth coordinate function x^0 in a relatively open neighborhood $\tilde{\Omega}$ of P, on the side of S that intersects Ω_1. We have thus obtained a "canonical coordinate system" for both metrics $\mathbf{g}, \tilde{\mathbf{g}}$. We then construct the map $\Psi : \tilde{\Omega} \to \tilde{\Omega}$ by identifying the coordinates for the two metrics $\mathbf{g}, \tilde{\mathbf{g}}$. By hypothesis we know that Ψ fixes S and $\Psi^*\tilde{\mathbf{g}} - \mathbf{g}$ vanishes to third order on S (since $\mathbf{g} = \tilde{\mathbf{g}}$ in Ω_2). They key feature is that we have that

$g_{0i} = g'_{0i} = 0$ for $i = 1, 2, 3$ and $g_{00} = g'_{00} = 1$ in $\tilde{\Omega}$. This allows us to derive the same system of equations as above. We note that as in [5], the function x^0 will be strongly pseudo-convex in a small enough neighborhood of P.

2.2 The Ionescu-Klainerman Carleman Estimate and our Main Proposition.

To state our main Proposition we must recall some results from [3]. The reader is referred to section 6 in that paper. Firstly recall the optical functions u_+, u_- defined near S. We condider the coordinate system $\{\frac{u_+ + u_-}{\sqrt{2}}, \frac{u_+ - u_-}{\sqrt{2}}, x^1, x^2\}$ defined over Ω and the function N^P defined over Ω via $N(P) := [u_+^2 + u_-^2 + (x^1)^2 + (x^2)^2]$;[7] recall also that that $B_{\epsilon^{10}}(P)$ stands for the set of points in Ω for which $(N^P)^2 \le \epsilon^{20}$. Then the Carleman weight function of Ionescu-Klainerman is: $f_\epsilon = ln\{\epsilon^{-1}(u_+ + \epsilon)(u_- + \epsilon) + \epsilon^{10} N^P\}$ (defined for some fixed small $\epsilon > 0$) from Lemma 6.2 in [3].

For any scalar-valued function Φ defined on $B_{\epsilon^{10}}$ we recall the weighted L^2-norm introduced in [3]:

$$||\Phi||_{L^2_\lambda} = \sqrt{\int_{B_{\epsilon^{10}}} \Phi^2 e^{-2\lambda \cdot f_\epsilon} dV_g}.$$

We introduce a cut-off function $\chi : \mathbb{R} \to [0, 1]$, defined to be smooth and supported in $[1/2, \infty]$, and equal to 1 in $[3/4, \infty]$. We then define $\eta_\epsilon : M \to [0, 1]$ to be $\eta_\epsilon(x) := 1 - \chi(N^P/\epsilon^{10}(x))$. (In the setting of Theorem 1.2 we can pick any cut-off function $\eta_\epsilon(x^0) = 1 - \chi(\frac{x^0}{\epsilon})$, for $\epsilon > 0$ small enough).

We denote by V_ϵ a generic function (independent of λ) which is supported in the set where $0 < \eta_\epsilon < 1$. We then recall the first Carleman estimate of Ionescu and Klainerman (see Lemma 6.2 in [3]): Setting $\phi = T_{abcd} \cdot \eta_\epsilon$ we derive that there exist a constants ϵ, C, λ_0 so that for every $\lambda > \lambda_0$:

$$\begin{aligned} &\lambda \cdot \sum_{a,b,c,d=0}^{3} ||T_{abcd} \cdot \eta_\epsilon||_{L^2_\lambda} + \sum_{a,b,c,d,e=0}^{3} ||\partial_e T_{abcd} \cdot \eta_\epsilon||_{L^2_\lambda} \le \\ &\frac{C}{\sqrt{\lambda}} \{ \sum_{a,b,c,d=0}^{3} ||\tilde{\Box}_g T_{abcd} \cdot \eta_\epsilon||_{L^2_\lambda} + ||V_\epsilon||_{L^2_\lambda} \}. \end{aligned} \tag{2.12}$$

Remark 7: In [3] this estimate is derived for functions ϕ in $C_0^\infty(B_{\epsilon^{10}})$; in fact, following the proof of this estimate in [3] we observe that that it holds for $\mathcal{C}^2$-functions ϕ which vanish on $\partial B_{\epsilon^{10}} \bigcap (\mathcal{H}^+ \bigcup \mathcal{H}^-)$ and vanish along with their first derivatives on $\partial B_{\epsilon^{10}} \setminus (\partial B_{\epsilon^0} \bigcap (\mathcal{H}^+ \bigcup \mathcal{H}^-)$. Thus we are allowed to set $\phi = T_{abcd} \cdot \eta_\epsilon$ and derive (2.12).

Remark 8: In the setting of Theorem 1.2 the analogous Carleman estimate for functions which are compactly supported in a small enough neighborhood of P in Ω_1 is classical, see [5]; in that setting the weight function can be chosen to be $f_\epsilon := x^0$, and the estimate holds for compactly supported functions in a neighborhood of P where the level sets of x^0 are strongly pseudo-convex, and for $\epsilon > 0$ small enough so that all the intersections $\{x^0 = \epsilon'\} \bigcap \Omega_1$, $\epsilon' < \epsilon$, are compact.

Now, using the equation (2.7) we derive that there exists a constant C' (independent of λ) so that:

[7] The function N^P is defined to be a distance function with respect to a Euclidean coordinate system around P, in the exterior region. For our purposes, we choose the coordinate system $\{\frac{u_+ + u_-}{\sqrt{2}}, \frac{u_+ - u_-}{\sqrt{2}}, x^1, x^2\}$ such that $N(P) := u_+^2 + u_-^2 + (x^1)^2 + (x^2)^2$.

$$||\widetilde{\Box_g}T_{abcd}\cdot\eta_\epsilon||_{L^2_\lambda}\le C'\{\sum_{a,b,c,d=0}^{3}||T_{abcd}||_{L^2_\lambda}+\sum_{a,b,c,d,e=0}^{3}||\partial_e T_{abcd}||_{L^2_\lambda}$$
$$+\sum_{a,b=0}^{3}||d_{ab}||_{L^2_\lambda}+\sum_{a,b,c=0}^{3}||\partial_c d_{ab}||_{L^2_\lambda}\sum_{a,b=0}^{3}+||D_{ab}||_{L^2_\lambda}+||V_\epsilon||_{L^2_\lambda}\}. \tag{2.13}$$

Our main Proposition is the following:

Proposition 2.1. *We claim that there exists a (universal) constant C and a number $\lambda_0>0$ so that for every $\lambda\ge\lambda_0$:*

$$\sum_{a,b=0}^{3}||d_{ab}||_{L^2_\lambda}\le\frac{C}{\sqrt{\lambda}}\{\sum_{a,b,c,d=0}^{3}||T_{abcd}\cdot\eta_\epsilon||_{L^2_\lambda}+||V_\epsilon||_{L^2_\lambda}\}, \tag{2.14}$$

$$\sum_{a,b,c=0}^{3}||\partial_c d_{ab}||_{L^2_\lambda}\le\frac{C}{\sqrt{\lambda}}\{\sum_{a,b,c,d=0}^{3}||T_{abcd}\cdot\eta_\epsilon||_{L^2_\lambda}+\sum_{a,b,c,d,e=0}^{3}||\partial_e T_{abcd}\cdot\eta_\epsilon||_{L^2_\lambda}+||V_\epsilon||_{L^2_\lambda}\}, \tag{2.15}$$

$$\sum_{a,b=0}^{3}||D_{ab}||_{L^2_\lambda}\le\frac{C}{\sqrt{\lambda}}\{\sum_{a,b,c,d=0}^{3}||T_{abcd}\cdot\eta_\epsilon||_{L^2_\lambda}+\sum_{a,b,c,d,e=0}^{3}||\partial_e T_{abcd}\cdot\eta_\epsilon||_{L^2_\lambda}$$
$$+\sum_{a,b,c,d=0}^{3}||\widetilde{\Box_g}T_{abcd}\cdot\eta_\epsilon||_{L^2_\lambda}+||V_\epsilon||_{L^2_\lambda}\}. \tag{2.16}$$

Let us check how the above will imply Theorem 1.1: Using the above three estimates and (2.13) we derive that there exists a $\lambda_0>0$ and a constant C' independent of λ so that for every $\lambda>\lambda_0$:

$$||\widetilde{\Box_g}T_{abcd}\cdot\eta_\epsilon||_{L^2_\lambda}\le C'\{\sum_{a,b,c,d=0}^{3}||T_{abcd}||_{L^2_\lambda}+\sum_{a,b,c,d,e=0}^{3}||\partial_e T_{abcd}||_{L^2_\lambda}+||V_\epsilon||_{L^2_\lambda}\}. \tag{2.17}$$

Thus, replacing the above into (2.12) we derive that for λ large enough:

$$\lambda\cdot\sum_{a,b,c,d=0}^{3}||T_{abcd}\cdot\eta_\epsilon||_{L^2_\lambda}+\sum_{a,b,c,d,e=0}^{3}||\partial_e T_{abcd}\cdot\eta_\epsilon||_{L^2_\lambda}\le\frac{C}{\sqrt{\lambda}}||V_\epsilon||_{L^2_\lambda}. \tag{2.18}$$

Now, the argument from page 35 in [3] implies that $T_{abcd}=0$ for all $\{a,b,c,d\}\in\{0,1,2,3,\}$ and for every $P\in B_{\epsilon^{40}}$.[8] Then, using (2.8) we derive that $d_{ab}=0$ for all $\{a,b\}\in\{0,1,2,3,\}$ in $B_{\epsilon^{40}}$. This shows that $\mathbf{g}=\mathbf{g}'$ in $B_{\epsilon^{40}}$. Similarly, in the setting of Theorem 1.2 we derive that $d_{ab}=0$ in the region $x^0\le\frac{\epsilon}{2}$.

[8]The point here is that the maximum value of the weight function $e^{-\lambda f_\epsilon}$ in the support of V_ϵ is bounded above by minimum value of $e^{-\lambda f_\epsilon}$ in $B_{\epsilon^{40}}$.

2.3 Proof of Proposition 2.1:

We now prove a main Lemma which will be very useful towards proving Proposition 2.1. Firstly let us make a note regarding the relation between the coordinate x^0 and the function u_+ introduced in [3]. Recall that $\mathcal{H}^+ = \{x^0 = 0\} = \{u_+ = 0\}$. Moreover there is a function ρ defined over $B_{\epsilon^{10}}$ so that for every point $P \in B_{\epsilon^{10}}$: $\frac{\partial}{\partial u_+} = \rho(P)\frac{\partial}{\partial x^0}$. There clearly exist numbers $0 < \mu \leq M$ so that for every $P \in B_{\epsilon^{10}}$, $0 < \mu \leq \rho(P) \leq M$. We now state our main claim:

Lemma 2.1. *Let ϕ be a function defined in $B_{\epsilon^{10}}$ which vanishes on $B_{\epsilon^{10}} \bigcap \mathcal{H}^+$.*[9] *Then we claim there exists a $C > 0$ so that for λ large enough:*

$$||e^{-\lambda \cdot f_\epsilon}\phi||_{L^2(B_{\epsilon^{10}})} \leq \frac{C}{\sqrt{\lambda}}||e^{-\lambda \cdot f_\epsilon}\partial_0\phi||_{L^2(B_{\epsilon^{10}})}. \tag{2.19}$$

Proof: Firstly a note about the volume form: The volume form dV_g is defined by a function $\omega(u_+, x^1, x^2, u_-)$ defined over $B_{\epsilon^{10}}$ so that:

$$dV_g = \omega(u_+, x^1, x^2, u_-)du_+ \wedge dx^1 \wedge dx^2 \wedge du_-.$$

Note that there exists constants μ', M' so that $0 < \mu' \leq \omega(P) \leq M'$ for every $P \in B_{\epsilon^{10}}$. By definition:

$$||e^{-\lambda \cdot f_\epsilon}\phi||^2_{L^2(B_{\epsilon^{10}})} = \int_{B_{\epsilon^{10}}} e^{-2\lambda \cdot f_\epsilon}\phi^2\omega du_+ \wedge dx^1 \wedge dx^2 \wedge du_-.$$

We observe that $e^{-2\lambda f_\epsilon} = [\epsilon^{-1}(u_+ + \epsilon)(u_- + \epsilon) + \epsilon^{10}(u_+^2 + u_-^2 + (x^1)^2 + (x^2)^2)]^{-2\lambda}$. Given fixed values for u_-, x^1, x^2, we set $w_\epsilon(t) := [\epsilon^{-1}(t + \epsilon)(u_- + \epsilon) + \epsilon^{10}(t^2 + u_-^2 + (x^1)^2 + (x^2)^2)]$.[10] Observe that for $(t, u_-, x^1, x^2) \in B_{\epsilon^{10}}$ we have a bound:

$$1 \leq \partial_t w_\epsilon(t) \leq 1 + 2\epsilon.$$

Now, given fixed vaues for x^1, x^2, u_- such that $(u_-)^2 + (x^1)^2 + (x^2)^2 \leq \epsilon^{20}$, we let $Max(u_-, x^1, x^2) := \sqrt{\epsilon^{20} - (u_-)^2 - (x^1)^2 - (x^2)^2}$; we claim:

$$\int_0^{Max(u_-,x^1,x^2)} (w_\epsilon(u_+))^{-2\lambda}\phi^2\omega du_+ \leq \frac{C}{\lambda}\int_0^{Max(u_-,x^1,x^2)} (w_\epsilon(u_+))^{-2\lambda}(\partial_0\phi)^2\omega du_+,$$

with the constant C independent of x^1, x^2, u_-. Clearly this will imply our claim.

We now prove the above. Some notational conventions: We write Max instead of $Max(u_-, x^1, x^2)$ for short. Moreover, when we write ∂_{u_+} we will be referring to differentiation with respect to the vector field $\frac{\partial}{\partial u_+}$, while ∂_0 will stand for differentiation with respect to the vector field $\frac{\partial}{\partial x^0}$. We derive:

$$\begin{aligned}
&\int_0^{Max} (w_\epsilon(u_+))^{-2\lambda}\phi^2\omega du_+ \leq M\int_0^{Max} (w_\epsilon(u_+))^{-2\lambda}\phi^2 du_+ = \int_0^{Max} (w_\epsilon(u_+))^{-2\lambda}(\int_0^{u_+} \partial_{u_+}\phi dt)^2 du_+ \\
&\leq M\int_0^{Max} [(w_\epsilon(u_+))^{-2\lambda}(\int_0^{u_+} (w_\epsilon(t))^{-2\lambda}(\partial_{u_+}\phi)^2 dt)(\int_0^{u_+} (w_\epsilon(t))^{2\lambda}dt)]du_+ \leq \\
&M'[\int_0^{Max} (w_\epsilon(u_+))^{-2\lambda}(\partial_{u_+}\phi)^2 du_+]\cdot\int_0^{Max} [(w_\epsilon(u_+))^{-2\lambda}\int_0^{u_+} (w_\epsilon(t))^{2\lambda}(\partial_t w_\epsilon(t))dt]du_+ \leq \\
&M'[\int_0^{Max} (w_\epsilon(u_+))^{-2\lambda}(\partial_{u_+}\phi)^2 du_+]\cdot\int_0^{Max} \frac{w_\epsilon(u_+)}{2\lambda+1}du_+ \leq \\
&\frac{M''}{\lambda}\int_0^{Max} (u_+ + \epsilon)^{-2\lambda}(\partial_{u_+}\phi)^2 du_+ \leq \frac{C}{\lambda}\int_0^{Max} (u_+ + \epsilon)^{-2\lambda}(\partial_0\phi)^2\omega du_+,
\end{aligned} \tag{2.20}$$

[9]Note that we are not requiring ϕ to be compactly supported in $B_{\epsilon^{10}}$.
[10]I.e. we are allowing the parameter u_+ to vary, and label it s.

QED. □

Remark 9: In the setting of Theorem 1.2 we can derive the exact same Lemma (with a gain of a factor $\frac{C}{\sqrt{\lambda}}$ in the RHS), with the classical weight function $e^{-\lambda\psi(x)}$,[11] in a small enough neighborhood of the point P.

We now use the above result to derive some estimates:

Lemma 2.2. *We claim that there exist constants C, λ_0 so that for every $\lambda > \lambda_0$:*

$$\sum_{a,b=0}^{3} ||d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda} \le \frac{C}{\sqrt{\lambda}}\{\sum_{a,b,c,d=0}^{3} ||T_{abcd}\cdot\eta_\epsilon||_{L^2_\lambda} + ||V_\epsilon||_{L^2_\lambda}\}, \tag{2.21}$$

$$\sum_{a,b,c=0}^{3} ||\partial_c d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda} \le \frac{C}{\sqrt{\lambda}}\{\sum_{a,b,c,d=0}^{3} ||T_{abcd}\cdot\eta_\epsilon||_{L^2_\lambda} + \sum_{a,b,c,d,e=0}^{3} ||\partial_e T_{abcd}\cdot\eta_\epsilon||_{L^2_\lambda} + ||V_\epsilon||_{L^2_\lambda}\}, \tag{2.22}$$

$$\sum_{a,b,c=0}^{3} ||\partial^{(2)}_{0c} d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda} \le \frac{C}{\sqrt{\lambda}}\{\sum_{a,b,c,d=0}^{3} ||T_{abcd}\cdot\eta_\epsilon||_{L^2_\lambda} + \sum_{a,b,c,d,e=0}^{3} ||\partial_e T_{abcd}\cdot\eta_\epsilon||_{L^2_\lambda} + ||V_\epsilon||_{L^2_\lambda}\}, \tag{2.23}$$

$$\sum_{a,b,c=0}^{3} ||\partial^{(3)}_{00c} d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda} \le \{\sum_{a,b,c,d=0}^{3} ||T_{abcd}\cdot\eta_\epsilon||_{L^2_\lambda} + \sum_{a,b,c,d,e=0}^{3} ||\partial_e T_{abcd}\cdot\eta_\epsilon||_{L^2_\lambda} + ||V_\epsilon||_{L^2_\lambda}\}. \tag{2.24}$$

Proof : We will prove (2.21). The other equations hold by the same argument. To prove (2.21) we repeatedly use the Lemma 2.1. By applying it once we derive:

$$\begin{aligned} &\sum_{a,b=0}^{3} ||d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda} \le \frac{C}{\sqrt{\lambda}}\sum_{a,b=0}^{3} ||\partial_0(d_{ab}\cdot\eta_\epsilon)||_{L^2_\lambda} \le \\ &\frac{C}{\sqrt{\lambda}}\sum_{a,b=0}^{3}\{||(\partial_0 d_{ab})\cdot\eta_\epsilon||_{L^2_\lambda} + \frac{C}{\sqrt{\lambda}}\sum_{a,b=0}^{3} ||d_{ab}\cdot\partial_0(\eta_\epsilon)||_{L^2_\lambda}\}. \end{aligned} \tag{2.25}$$

Analogously we derive:

$$\begin{aligned} &\sum_{a,b=0}^{3} ||(\partial_0 d_{ab})\cdot\eta_\epsilon||_{L^2_\lambda} \le \frac{C}{\sqrt{\lambda}}\sum_{a,b=0}^{3} ||\partial_0[(\partial_0 d_{ab})\cdot\eta_\epsilon]||_{L^2_\lambda} \le \\ &\frac{C}{\sqrt{\lambda}}\sum_{a,b=0}^{3}\{||(\partial^{(2)}_{00} d_{ab})\cdot\eta_\epsilon||_{L^2_\lambda} + \sum_{a,b=0}^{3} ||\partial_0 d_{ab}\cdot\partial_0(\eta_\epsilon)||_{L^2_\lambda}\} \le \\ &\frac{C}{\sqrt{\lambda}}\sum_{a,b=0}^{3}\{||2(T_{0ab0} - L^{yu}_{ab}\partial_0 d_{yu} - L'^{yu}_{ab} d_{yu})\cdot\eta_\epsilon||_{L^2_\lambda} + \sum_{a,b=0}^{3} ||V_\epsilon||_{L^2_\lambda}\} \le \\ &\frac{C}{\sqrt{\lambda}}\sum_{a,b=0}^{3}\{||T_{0ab0}\cdot\eta_\epsilon||_{L^2_\lambda} + \sum_{a,b=0}^{3} ||\partial_0 d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda} + \sum_{a,b=0}^{3} ||d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda} + ||V_\epsilon||_{L^2_\lambda}\}. \end{aligned} \tag{2.26}$$

Now, we can control the term $\sum_{a,b=0}^{3} ||d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}$ in the RHS using (2.25); furthermore, for λ large enough the term $\frac{C}{\sqrt{\lambda}}\sum_{a,b=0}^{3} ||\partial_0 d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}$ in the RHS can be absorbed into the LHS and thus we derive the equation:

[11] Here $\psi(x) = x^0$, in the notation of remark 5.

$$\sum_{a,b=0}^{3} ||(\partial_0 d_{ab})\cdot\eta_\epsilon||_{L^2_\lambda} \le \frac{C}{\sqrt{\lambda}}\{\sum_{a,b=0}^{3}||T_{0ab0}\cdot\eta_\epsilon||_{L^2_\lambda} + \sum_{a,b=0}^{3}||V_\epsilon||_{L^2_\lambda}\}. \tag{2.27}$$

Now, combining (2.27) with (2.25) we also derive (2.21). Equations (2.22), (2.23), (2.24) follow by a straightforward adaptation of this argument. □

We now claim a more involved Lemma.

Lemma 2.3. *We claim that:*

$$\sum_{a,b,c=0}^{3}||D_{ab}\cdot\eta_\epsilon||_{L^2_\lambda} \le \frac{C}{\sqrt{\lambda}}\{\sum_{a,b,c,d=0}^{3}||T_{abcd}\cdot\eta_\epsilon||_{L^2_\lambda} + \sum_{a,b,c,d,e=0}^{3}||\partial_e T_{abcd}\cdot\eta_\epsilon||_{L^2_\lambda} + \sum_{a,b,c,d=0}^{3}||\widetilde{\Box_g}T_{abcd}\cdot\eta_\epsilon||_{L^2_\lambda} + ||V_\epsilon||_{L^2_\lambda}\}. \tag{2.28}$$

Note that in view of Lemma 2.2, this Lemma will imply the main Proposition 2.1, and hence also Theorem 1.1.

Proof of Lemma 2.3: We will prove the above in two pieces. Firstly, recall that $\mathbf{D_{(ab)}}$ stands for the symmetric part of the 2-tensor $\mathbf{D_{ab}}$ (so $\mathbf{D_{(ab)}} = \frac{\mathbf{1}}{\mathbf{2}}[\mathbf{D_{ab}}+\mathbf{D_{ba}}]$) and $\mathbf{D_{[ab]}}$ stands for the antisymmetric part (so $\mathbf{D_{[ab]}} = \frac{\mathbf{1}}{\mathbf{2}}[\mathbf{D_{ab}}-\mathbf{D_{ba}}]$). We will prove (2.28) separately for the symmetric part $\mathbf{D_{(ab)}}$ and the antisymmetric part $\mathbf{D_{[ab]}}$ of $\mathbf{D_{ab}}$. Specifically we will prove:

$$\sum_{a,b=0}^{3}||D_{(ab)}\cdot\eta_\epsilon||_{L^2_\lambda} \le \frac{C}{\sqrt{\lambda}}\{\sum_{a,b,c,d=0}^{3}||T_{abcd}\cdot\eta_\epsilon||_{L^2_\lambda} + \sum_{a,b,c,d,e=0}^{3}||\partial_e T_{abcd}\cdot\eta_\epsilon||_{L^2_\lambda} + \sum_{a,b,c,d=0}^{3}||\widetilde{\Box_g}T_{abcd}\cdot\eta_\epsilon||_{L^2_\lambda} + ||V_\epsilon||_{L^2_\lambda}\}, \tag{2.29}$$

$$\sum_{a,b=0}^{3}||D_{[ab]}\cdot\eta_\epsilon||_{L^2_\lambda} \le \frac{C}{\sqrt{\lambda}}\{\sum_{a,b,c,d=0}^{3}||T_{abcd}\cdot\eta_\epsilon||_{L^2_\lambda} + \sum_{a,b,c,d,e=0}^{3}||\partial_e T_{abcd}\cdot\eta_\epsilon||_{L^2_\lambda} + \sum_{a,b,c,d=0}^{3}||\widetilde{\Box_g}T_{abcd}\cdot\eta_\epsilon||_{L^2_\lambda} + ||V_\epsilon||_{L^2_\lambda}\}. \tag{2.30}$$

Proof of (2.29): Recall that $D_{(ab)} = -\frac{1}{2}\widetilde{\Box_g}d_{ab}(= -\frac{1}{2}g^{yu}\partial^{(2)}_{yu}d_{ab})$. Let us also make an important observation: Since we have $g_{0b} = g_{b0} = 0$ for $b = 0,1,2$ and $g_{03} = g_{30} = 1$, we derive that $g^{3b} = g^{b3} = 0$ for $b = 0,1,2$ and $g^{03} = g^{30} = 1$. We now claim two useful estimates which we will prove later. First useful estimate:

$$\sum_{a,b=0}^{3}||(\partial_0 g^{yu})\partial^{(2)}_{yu}d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda} \le C\{\sum_{a,b=0}^{3}||\widetilde{\Box_g}d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda} + \sum_{a,b,c,d=0}^{3}||T_{abcd}\cdot\eta_\epsilon||_{L^2_\lambda} + \sum_{a,b,c,d,e=0}^{3}||\partial_e T_{abcd}\cdot\eta_\epsilon||_{L^2_\lambda} + ||V_\epsilon||_{L^2_\lambda}\}. \tag{2.31}$$

Second estimate:

$$\sum_{a,b=0}^{3}||(\partial_0 g^{yu})\partial^{(3)}_{yu0}d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}\le C\{\sum_{a,b=0}^{3}||\widetilde{\Box_g}\partial_0 d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}+\sum_{a,b,c,d=0}^{3}||T_{abcd}\cdot\eta_\epsilon||_{L^2_\lambda}$$
$$+\sum_{a,b,c,d,e=0}^{3}||\partial_e T_{abcd}\cdot\eta_\epsilon||_{L^2_\lambda}+||V_\epsilon||_{L^2_\lambda}\}. \tag{2.32}$$

Let us check how the two equations (2.31), (2.32) will imply (2.29). We start by applying Lemma 2.1:

$$\sum_{a,b=0}^{3}||\widetilde{\Box_g}d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}\le\frac{C}{\sqrt{\lambda}}[\sum_{a,b=0}^{3}||\partial_0[\widetilde{\Box_g}d_{ab}\cdot\eta_\epsilon]||_{L^2_\lambda}]\le$$
$$\frac{C}{\sqrt{\lambda}}\{\sum_{a,b=0}^{3}||g^{yu}\partial^{(3)}_{yu0}d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}+\sum_{a,b=0}^{3}||(\partial_0 g^{yu})\partial^{(2)}_{yu}d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}+||V_\epsilon||_{L^2_\lambda}\}. \tag{2.33}$$

Therefore, using (2.31) and taking λ large enough we derive:

$$\sum_{a,b=0}^{3}||\widetilde{\Box_g}d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}\le\frac{C}{\sqrt{\lambda}}\{\sum_{a,b=0}^{3}||g^{yu}\partial^{(3)}_{yu0}d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}$$
$$+\sum_{a,b,c,d=0}^{3}||T_{abcd}\cdot\eta_\epsilon||_{L^2_\lambda}+\sum_{a,b,c,d,e=0}^{3}||\partial_e T_{abcd}\cdot\eta_\epsilon||_{L^2_\lambda}+||V_\epsilon||_{L^2_\lambda}\}. \tag{2.34}$$

Again applying Lemma 2.1 we derive:

$$\sum_{a,b=0}^{3}||g^{yu}\partial^{(3)}_{yu0}d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}\le\frac{C}{\sqrt{\lambda}}\sum_{a,b=0}^{3}||\partial_0[g^{yu}\partial^{(3)}_{yu0}d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}]\le$$
$$\frac{C}{\sqrt{\lambda}}\{\sum_{a,b=0}^{3}||g^{yu}\partial^{(4)}_{yu00}d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}+\sum_{a,b=0}^{3}||(\partial_0 g^{yu})\partial^{(3)}_{yu0}d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}+||V_\epsilon||_{L^2_\lambda}\}\le$$
$$\frac{C}{\sqrt{\lambda}}\{\sum_{a,b=0}^{3}||2g^{yu}\partial^{(2)}_{yu}(T_{0ab0}-L^{yu}_{ab}\partial_0 d_{yu}-L'^{yu}_{ab}d_{yu})\cdot\eta_\epsilon||_{L^2_\lambda}+$$
$$\sum_{a,b=0}^{3}||(\partial_0 g^{yu})\partial^{(3)}_{yu0}d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}+||V_\epsilon||_{L^2_\lambda}\}\le\frac{C}{\sqrt{\lambda}}\{\sum_{a,b=0}^{3}||2g^{yu}\partial^{(2)}_{yu}T_{0ab0}\cdot\eta_\epsilon||_{L^2_\lambda}+$$
$$\sum_{a,b=0}^{3}||g^{yu}\partial^{(3)}_{yu0}d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}+\sum_{a,b=0}^{3}||g^{yu}\partial^{(2)}_{yu}d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}+$$
$$\sum_{a,b,c=0}^{3}||\partial^{(2)}_{0c}d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}+\sum_{a,b,c=0}^{3}||\partial_c d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}+\sum_{a,b=0}^{3}||d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}+$$
$$\sum_{a,b=0}^{3}||(\partial_0 g^{yu})\partial^{(3)}_{yu0}d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}+||V_\epsilon||_{L^2_\lambda}\}. \tag{2.35}$$

Now, the terms $\sum_{a,b,c=0}^{3}||\partial^{(2)}_{0c}d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}$, $\sum_{a,b,c=0}^{3}||\partial_c d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}$, $\sum_{a,b=0}^{3}||d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}$ can be controlled by virtue of Lemma 2.2. Then, combining (2.33) and (2.35) to also control the term $\sum_{a,b=0}^{3}||g^{yu}\partial^{(2)}_{yu}d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}$ $(=\sum_{a,b=0}^{3}||\widetilde{\Box_g}d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda})$ and absorb it into the LHS we derive:

$$\sum_{a,b=0}^{3}||g^{yu}\partial^{(3)}_{yu0}d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}\le\frac{C}{\sqrt{\lambda}}\{\sum_{a,b=0}^{3}||g^{yu}\partial^{(3)}_{yu0}d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}+\sum_{a,b=0}^{3}||\widetilde{\Box_g}T_{0ab0}\cdot\eta_\epsilon||_{L^2_\lambda}$$
$$+\sum_{a,b,c,d=0}^{3}||T_{abcd}\cdot\eta_\epsilon||_{L^2_\lambda}+\sum_{a,b,c,d,e=0}^{3}||\partial_eT_{abcd}\cdot\eta_\epsilon||_{L^2_\lambda}+||V_\epsilon||_{L^2_\lambda}\}. \tag{2.36}$$

Therefore for λ large enough we can absorb the term $||g^{yu}\partial^{(3)}_{yu0}d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}$ from the RHS into the LHS; then, substituting in this estimate into the (2.34) we derive (2.29), subject to proving (2.31), (2.32). We now prove these two equations:

Proof of (2.31): The key in the proof of this estimate is the fact that $\partial_0g^{3b}=0$ for $b=0,1,2,3$,[12] and that we have already controlled the weighted L^2-norms of the functions $\partial^{(2)}_{0c}d_{ab}$. With these observations the desired estimate (2.31) will readily be reduced to elliptic estimates on the level sets of the Carleman weight function f_ϵ: Since $g^{3b}=0$ for $b=1,2,3$, we derive that:

$$|(\partial_0g^{cd})\partial^{(2)}_{cd}d_{ab}|^2\le\sum_{c,d=1}^{2}|(\partial_0g^{cd})\partial^{(2)}_{cd}d_{ab}|^2+2\sum_{r=0}^{3}|(\partial_0g^{r0})\partial^{(2)}_{0r}d_{ab}|^2.$$

Let us make a few notes that will be useful further down. Firstly recall that the Fermi coordinate system we have introduced defines the coordinate vector fields: Let $\{X^0,X^1,X^2,X^3\}$ be the vector fields $\{\frac{\partial}{\partial x^0},\frac{\partial}{\partial x^1},\frac{\partial}{\partial x^2},\frac{\partial}{\partial x^3}\}$. Then when we give values $_{0,1,2,3}$ to the lower indices $_{\alpha,\beta}$ in $\partial_{\alpha\beta}$ those values correspond to these vector fields X^0,X^1,X^2,X^3 above. In that notation, recall the "rough wave operator":

$$\widetilde{\Box_g}=\sum_{c,d=1}^{2}g^{cd}\partial^{(2)}_{cd}+2\sum_{c=1}^{3}g^{0r}\partial^{(2)}_{0r}+g^{00}\partial^{(2)}_{00}. \tag{2.37}$$

We will consider the operator $\sum_{c,d=1}^{2}g^{cd}\partial^{(2)}_{cd}$ separately and denote it by $\tilde{\Delta}_{g,1}$; we call it the "first rough Laplacian". We will now introduce a second basis for $T(B_{\epsilon^{10}})$ which will be useful: Recall the Carleman weight function f_ϵ. By virtue of the form of f_ϵ and the fact that $\{x^3=C\}=\{u_-=C\}$, we derive that there exist smooth functions ρ^1,ρ^2 defined over $B_{\epsilon^{10}}$ such that the vector fields $\tilde{X}^1=X^1+\rho^1X^0$ and $\tilde{X}^2=X^2+\rho^2X^0$ are tangent to the level sets of f_ϵ.[13] Thus, we obtain a new frame at each point $P\in B_{\epsilon^{10}}$, $\{\tilde{X}^0=X^0,\tilde{X}^1,\tilde{X}^2,\tilde{X}^3=X^3\}$. When we refer to components of tensors with respect to this new frame we will use indices with tildes e.g. $_{\tilde{0}},{}^{\tilde{2}}$ etc. We observe that $g_{\tilde{0}\tilde{a}}=g_{\tilde{a}\tilde{0}}=0$ for $_a={}_{0,1,2}$ and also $g_{\tilde{0}\tilde{3}}=g_{\tilde{3}\tilde{0}}=1$ thus again we derive that $g^{\tilde{3}\tilde{a}}=g^{\tilde{a}\tilde{3}}=0$ for ${}^{\tilde{a}}={}^{\tilde{0}},{}^{\tilde{1}},{}^{\tilde{2}}$ and $g^{\tilde{3}\tilde{0}}=g^{\tilde{0}\tilde{3}}=1$. In fact we observe that $g^{\tilde{c}\tilde{d}}=g^{cd}$ for ${}^c,{}^d={}^1,{}^2$. We now define the "second rough wave operator" with respect to this frame:

$$\widetilde{\Box_{g,2}}=\sum_{\tilde{c},\tilde{d}=1}^{2}g^{\tilde{c}\tilde{d}}\partial^{(2)}_{\tilde{c}\tilde{d}}+\sum_{\tilde{r}=\tilde{1}}^{3}g^{\tilde{0}\tilde{r}}\partial^{(2)}_{\tilde{0}\tilde{r}}+g^{\tilde{0}\tilde{0}}\partial^{(2)}_{\tilde{0}\tilde{0}}. \tag{2.38}$$

Observe that:

[12]This remains true in the setting of Theorem 1.2.

[13]In the setting of Theorem 1.2 there is no need to modify the vector fieilds X^1,X^2–these are already tangent to the level sets of the Carleman weight function.

$$\widetilde{\Box_g} = \widetilde{\Box_{g,2}} + \sum_{i=0}^{3} L^i \partial_i. \tag{2.39}$$

(So the 2nd term in the RHS is a generic linear combination of first order operators with $\mathcal{C}^3$ coefficients). We again separately consider the operator $\sum_{\tilde{c},\tilde{d}=1}^{2} g^{\tilde{c}\tilde{d}}\partial^{(2)}_{\tilde{c}\tilde{d}}$ which we will call the *second rough Laplacian* and denote it by $\tilde{\Delta}_{g,2}$. Observe that:

$$\tilde{\Delta}_g = \tilde{\Delta}_{g,2} + \sum_{i=0}^{2} R^i \partial_i + \sum_{i=0}^{2} R'^i \partial^{(2)}_{0i}. \tag{2.40}$$

It then follows straightforwardly that there exists a constant $C > 0$ such that:

$$[\sum_{c,d=1}^{2} |(\partial_0 g^{cd})\partial^{(2)}_{cd} d_{ab}|]^2 \le C[\sum_{\tilde{c},\tilde{d},\tilde{e},\tilde{r}=1}^{2} g^{\tilde{c}\tilde{d}} g^{\tilde{e}\tilde{r}} \partial^{(2)}_{\tilde{c}\tilde{e}} d_{ab} \partial^{(2)}_{\tilde{d}\tilde{r}} d_{ab} + \sum_{c=0}^{2}[(\partial_c d_{ab})^2 + (\partial^{(2)}_{0c} d_{ab})^2]]. \tag{2.41}$$

Using the above we can now prove (2.31). Directly applying the above we obtain:

$$\sum_{a,b=0}^{3} ||(\partial_0 g^{yu})\partial^{(2)}_{yu} d_{ab} \cdot \eta_\epsilon||^2_{L^2_\lambda} \le C' \cdot \sum_{c,d=1}^{2} \int_{B_{\epsilon^{10}}} [g^{\tilde{c}\tilde{d}} g^{\tilde{e}\tilde{r}} \partial^{(2)}_{\tilde{c}\tilde{e}} d_{ab} \partial^{(2)}_{\tilde{d}\tilde{r}} d_{ab}] \cdot e^{-2\lambda f_\epsilon} \eta^2_\epsilon dV_g$$
$$+ \sum_{c=0}^{3} ||\partial_c d_{ab} \cdot \eta_\epsilon||^2_{L^2_\lambda} + \sum_{c=0}^{3} ||\partial^{(2)}_{0c} d_{ab} \cdot \eta_\epsilon||^2_{L^2_\lambda}. \tag{2.42}$$

(Here dV_g is the volume form for the metric g on $B_{\epsilon^{10}}$).

Now, the trick is to estimate the term $\sum_{c,d=1}^{2} \int_{B_{\epsilon^{10}}} [g^{\tilde{c}\tilde{d}} g^{\tilde{e}\tilde{r}} \partial^{(2)}_{\tilde{c}\tilde{e}} d_{ab} \partial^{(2)}_{\tilde{d}\tilde{r}} d_{ab}] \cdot e^{-2\lambda f_\epsilon} \eta_\epsilon dV_g$ in the first line of the RHS. We integrate by parts twice with respect to $\partial_{\tilde{d}}$ and then $\partial_{\tilde{e}}$; since the vector fields $\tilde{X}^1, \tilde{X}^2$ (with respect to which we are integrating by parts) are *tangent* to the level sets of f_ϵ, we will *not* bring out derivatives of the factor $e^{-2\lambda f_\epsilon}$. We derive:

$$\begin{aligned}
&\sum_{\tilde{c},\tilde{d},\tilde{e},\tilde{r}=1}^{\tilde{2}} \int_{B_{\epsilon^{10}}} [g^{\tilde{c}\tilde{d}} g^{\tilde{e}\tilde{r}} \partial^{(2)}_{\tilde{c}\tilde{e}} d_{ab} \partial^{(2)}_{\tilde{d}\tilde{r}} d_{ab}] \cdot e^{-2\lambda f_\epsilon} \cdot \eta^2_\epsilon dV_g = \\
&\int_{B_{\epsilon^{10}}} [\tilde{\Delta}_{g,2} d_{ab}]^2 \cdot e^{-2\lambda f_\epsilon} \eta^2_\epsilon dV_g + \int_{B_{\epsilon^{10}}} L^{yu} \partial_y d_{ab} \partial_u d_{ab} \cdot e^{-2\lambda f_\epsilon} \eta^2_\epsilon dV_g + \\
&\int_{B_{\epsilon^{10}}} L^y \partial_y d_{ab} \tilde{\Delta}_{g,2} d_{ab} \cdot e^{-2\lambda f_\epsilon} \eta^2_\epsilon dV_g + \int_{B_{\epsilon^{10}}} V_\epsilon e^{-2\lambda f_\epsilon} \eta^2_\epsilon dV_g.
\end{aligned} \tag{2.43}$$

Thus applying Cauchy-Schwartz to the above we deduce:

$$\sum_{\tilde{c},\tilde{d},\tilde{e},\tilde{r}=1}^{\tilde{2}} \int_{B_{\epsilon^{10}}} [g^{\tilde{c}\tilde{d}} g^{\tilde{e}\tilde{r}} \partial^{(2)}_{\tilde{c}\tilde{e}} d_{ab} \partial^{(2)}_{\tilde{d}\tilde{r}} d_{ab}] \cdot e^{-2\lambda f_\epsilon} \eta^2_\epsilon dV_g \le C\{\int_{B_{\epsilon^{10}}} [\tilde{\Delta}_{g,2} d_{ab}]^2 \cdot e^{-2\lambda f_\epsilon} \eta^2_\epsilon dV_g$$
$$+ \sum_{c=0}^{2} \int_{B_{\epsilon^{10}}} (\partial_c d_{ab})^2 e^{-2\lambda f_\epsilon} \eta^2_\epsilon dV_g + \int_{S_{u_+,u_-}} V_\epsilon e^{-2\lambda f_\epsilon} \eta^2_\epsilon dV_g. \tag{2.44}$$

Replacing the above into (2.42) we derive:

$$\sum_{a,b=0}^{3}||(\partial_0 g^{yu})\partial^{(2)}_{yu}d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}\le C\{\sum_{a,b=0}^{3}||\tilde{\Delta}_{g,2}d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}+ \\ \sum_{c=0}^{3}||\partial^{(2)}_{c0}d_{ab}||_{L^2_\lambda}+\sum_{c=0}^{3}||\partial_c d_{ab}||_{L^2_\lambda}+||V_\epsilon||_{L^2_\lambda}\}. \tag{2.45}$$

Now, using the formulas (2.37)–(2.40) we derive that $|\tilde{\Delta}_{g,2}d_{ab}|=|\widetilde{\Box_g}d_{ab}+\sum_{r=0}^{3}C^r\partial^{(2)}_{0r}d_{ab}+\sum_{r=0}^{3}C'^r\partial_r d_{ab}|$ for some $\mathcal{C}^2$ functions C^r, C'^r. Thus substituting the above into (2.46) we derive that:

$$\sum_{a,b=0}^{3}||(\partial_0 g^{yu})\partial^{(2)}_{yu}d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}\le C\{\sum_{a,b=0}^{3}||\widetilde{\Box_g}d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}+ \\ \sum_{c=0}^{3}||\partial^{(2)}_{c0}d_{ab}||_{L^2_\lambda}+\sum_{c=0}^{3}||\partial_c d_{ab}||_{L^2_\lambda}+||V_\epsilon||_{L^2_\lambda}\}. \tag{2.46}$$

Thus, combining the above with Lemma 2.2 we derive our claim.

Proof of (2.32): The proof of this claim is very much in the spirit of the previous one. We again use the formulas (2.37)–(2.40) to derive the analogue of (2.41):

$$[\sum_{c,d=1}^{2}|(\partial_0 g^{cd})\partial^{(3)}_{cd0}d_{ab}|]^2\le C\{\sum_{\tilde{c},\tilde{d},\tilde{e},\tilde{r}=\tilde{1}}^{\tilde{2}}g^{\tilde{c}\tilde{d}}g^{\tilde{e}\tilde{r}}\partial^{(3)}_{\tilde{c}\tilde{e}\tilde{0}}d_{ab}\partial^{(3)}_{\tilde{d}\tilde{r}\tilde{0}}d_{ab}+\sum_{c=0}^{3}[(\partial_c d_{ab})^2+(\partial^{(2)}_{c0}d_{ab})^2(\partial^{(3)}_{c00}d_{ab})^2]\}. \tag{2.47}$$

Thus, using the above we derive an analogue of (2.42):

$$\sum_{a,b=0}^{3}||(\partial_0 g^{yu})\partial^{(3)}_{yu0}d_{ab}\cdot\eta_\epsilon||^2_{L^2_\lambda}\le C'\cdot\{\sum_{\tilde{c},\tilde{d},\tilde{e},\tilde{r}=\tilde{1}}^{\tilde{2}}\int_{B_{\epsilon 10}}[g^{\tilde{c}\tilde{d}}g^{\tilde{e}\tilde{r}}\partial^{(3)}_{\tilde{c}\tilde{e}\tilde{0}}d_{ab}\partial^{(3)}_{\tilde{d}\tilde{r}\tilde{0}}d_{ab}]\cdot e^{-2\lambda f_\epsilon}\eta^2_\epsilon dV_g \\ +\sum_{c=0}^{3}[||\partial^{(2)}_{c0}d_{ab}\cdot\eta_\epsilon||^2_{L^2_\lambda}+||\partial^{(3)}_{c00}d_{ab}\cdot\eta_\epsilon||^2_{L^2_\lambda}+||\partial^{(2)}_{c0}d_{ab}\cdot\eta_\epsilon||^2_{L^2_\lambda}]\}= \\ C'\cdot\sum_{c,d=1}^{2}\int_{B_{\epsilon 10}}[g^{\tilde{c}\tilde{d}}g^{\tilde{e}\tilde{r}}\partial^{(3)}_{\tilde{c}\tilde{e}\tilde{0}}d_{ab}\partial^{(3)}_{\tilde{d}\tilde{r}\tilde{0}}d_{ab}]\cdot e^{-2\lambda f_\epsilon}\eta_\epsilon dV_g \\ +\sum_{c=0}^{3}[||\partial^{(2)}_{c0}d_{ab}\cdot\eta_\epsilon||^2_{L^2_\lambda}+||\partial_c d_{ab}\cdot\eta_\epsilon||^2_{L^2_\lambda}+||\partial^{(3)}_{c00}d_{ab}\cdot\eta_\epsilon||^2_{L^2_\lambda}]. \tag{2.48}$$

Then, performing the same integration by parts as for (2.44) we finally derive the analogue of (2.46):

$$\sum_{a,b=0}^{3}||(\partial_0 g^{yu})\partial^{(3)}_{yu0}d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}\le C'\{\sum_{a,b=0}^{3}||\widetilde{\Box_g}\partial_0 d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}+ \\ +\sum_{c=0}^{3}[||\partial^{(2)}_{c0}d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}+||\partial_c d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}+||\partial^{(3)}_{c00}d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}]+||V_\epsilon||_{L^2_\lambda}\}. \tag{2.49}$$

Then, again invoking Lemma 2.2 we derive our claim. □

Proof of (2.30): Again we start by applying Lemma 2.1, and then use the equation (2.10):

$$\sum_{a,b=0}^{3}||D_{[ab]}\cdot\eta_\epsilon||_{L^2_\lambda}\le\frac{C}{\sqrt{\lambda}}\sum_{a,b=0}^{3}||\partial_0 D_{[ab]}\cdot\eta_\epsilon||_{L^2_\lambda}+||V_\epsilon||_{L^2_\lambda}\le$$
$$\frac{C}{\sqrt{\lambda}}\{\sum_{a,b=0}^{3}||d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}+\sum_{a,b,c=0}^{3}||\partial_c d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}+\sum_{a,b,c,d=0}^{3}||T_{abcd}\cdot\eta_\epsilon||_{L^2_\lambda}+\sum_{a,b,c=0}^{3}||\partial_{0c}d_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}+$$
$$\sum_{a,b=0}^{3}||D_{ab}\cdot\eta_\epsilon||_{L^2_\lambda}+\sum_{a,b=0}^{3}||\sum_{i,j=1}^{2}c^{ij}\partial_i G_{j[a,b]}\cdot\eta_\epsilon||_{L^2_\lambda}+||V_\epsilon||_{L^2_\lambda}\}. \tag{2.50}$$

Now, observe that all the terms in the second line can be controlled by virtue of Lemma 2.2. Furthermore, we straightforwardly obtain:

$$\sum_{a,b=0}^{3}||D_{ab}||_{L^2_\lambda}\le\sum_{a,b=0}^{3}||D_{(ab)}||_{L^2_\lambda}+\sum_{a,b=0}^{3}||D_{[ab]}||_{L^2_\lambda},$$

and now the term $\sum_{a,b=0}^{3}||D_{[ab]}||_{L^2_\lambda}$ can be absorbed into the LHS (when λ is large enough), while we have already controlled $\sum_{a,b=0}^{3}||D_{(ab)}||_{L^2_\lambda}$ by virtue of equation (2.29). Thus, matters are reduced to controlling the term $\sum_{a,b=0}^{3}||\sum_{i,j=1}^{2}c^{ij}\partial_i G_{j[a,b]}||_{L^2_\lambda}$.

To do this we again will use the vector fields $\tilde{X}^1,\tilde{X}^2$ defined above (see the discussion after (2.37)), and we will evaluate the tensor $\mathbf{G_{ab,c}}$ against those vector fields.

Using this observation we can again derive that:

$$|\sum_{i,j=1}^{2}c^{ij}\partial_i G_{j[a,b]}|^2\le C\cdot|\sum_{\tilde{i},\tilde{j}=1}^{2}\sum_{\tilde{z},\tilde{q}=0}^{2}g^{\tilde{i}\tilde{j}}g^{\tilde{z}\tilde{q}}\partial_{\tilde{i}}G_{\tilde{z}[a,b]}\partial_{\tilde{j}}G_{\tilde{q}[a,b]}+\sum_{c=0}^{3}[|\partial_0 G_{c[a,b]}|^2+|\partial_c G_{0[a,b]}|^2].$$

Now, we straightforwardly see that $\sum_{a,b,c=0}^{3}|\partial_0 G_{c[a,b]}|+|\partial_c G_{0[a,b]}|\le C\sum_{a,b,c=0}^{3}|\partial^{(2)}_{0c}d_{ab}|$. Thus matters are reduced to controlling $\sum_{a,b=0}^{3}\int_{B_{\epsilon^{10}}}\sum_{\tilde{i},\tilde{j}=1}^{2}\sum_{\tilde{z},\tilde{q}=0}^{2}g^{\tilde{i}\tilde{j}}g^{\tilde{z}\tilde{q}}\partial_{\tilde{i}}G_{\tilde{z}[a,b]}\partial_{\tilde{j}}G_{\tilde{q}[a,b]}\cdot\eta^2_\epsilon dV_g$. We will prove:

$$\sum_{a,b=0}^{3}\int_{B_{\epsilon^{10}}}\sum_{\tilde{i},\tilde{j}=1}^{2}\sum_{\tilde{z},\tilde{q}=0}^{2}g^{\tilde{i}\tilde{j}}g^{\tilde{z}\tilde{q}}\partial_{\tilde{i}}G_{\tilde{z}[a,b]}\partial_{\tilde{j}}G_{\tilde{q}[a,b]}\cdot\eta^2_\epsilon dV_g\le$$
$$C\{\sum_{a,b=0}^{3}||D_{[ab]}\cdot\eta_\epsilon||^2_{L^2_\lambda}+\sum_{a,b,c=0}^{3}||\partial_c d_{[ab]}\cdot\eta_\epsilon||^2_{L^2_\lambda}+\sum_{a,b,c=0}^{3}||\partial^{(2)}_{0c}d_{[ab]}\cdot\eta_\epsilon||^2_{L^2_\lambda}\}. \tag{2.51}$$

We do this by first commuting indices using the relation:

$$\partial_{\tilde{\alpha}}G_{\tilde{\beta}i,j}-\partial_{\tilde{\beta}}G_{\tilde{\alpha}i,j}=T_{\tilde{\alpha}\tilde{\beta}ij}+L^{yur}_{\tilde{\alpha}\tilde{\beta}ij}G_{yu,r}+L'^{yu}_{\tilde{\alpha}\tilde{\beta}ij}d_{yu},$$

and then integrating by parts. Explicitly we derive:

$$\sum_{a,b=0}^{3}\int_{B_{\epsilon 10}}\sum_{\tilde{i},\tilde{j}=1}^{2}\sum_{\tilde{z},\tilde{q}=0}^{2} g^{\tilde{i}\tilde{j}}g^{\tilde{z}\tilde{q}}\partial_{\tilde{i}}G_{\tilde{z}[a,b]}\partial_{\tilde{j}}G_{\tilde{q}[a,b]}e^{-2\lambda f_\epsilon}\cdot\eta_\epsilon^2 dV_g =$$
$$\sum_{a,b=0}^{3}\int_{B_{\epsilon 10}}\sum_{\tilde{i},\tilde{j}=1}^{2}\sum_{\tilde{z},\tilde{q}=0}^{2} g^{\tilde{i}\tilde{j}}g^{\tilde{z}\tilde{q}}\partial_{\tilde{z}}G_{\tilde{i}[a,b]}\partial_{\tilde{j}}G_{\tilde{q}[a,b]}e^{-2\lambda f_\epsilon}\cdot\eta_\epsilon^2 dV_g +$$
$$\sum_{a,b=0}^{3}\int_{B_{\epsilon 10}}\sum_{\tilde{i},\tilde{j}=1}^{2}\sum_{\tilde{z},\tilde{q}=0}^{2} g^{\tilde{i}\tilde{j}}g^{\tilde{z}\tilde{q}}T_{\tilde{i}\tilde{z}ab}\partial_{\tilde{j}}G_{\tilde{q}[a,b]}e^{-2\lambda f_\epsilon}\cdot\eta_\epsilon^2 dV_g + \tag{2.52}$$
$$\sum_{a,b=0}^{3}\int_{B_{\epsilon 10}}\sum_{\tilde{i},\tilde{j}=1}^{2}\sum_{\tilde{z},\tilde{q}=0}^{2} g^{\tilde{i}\tilde{j}}g^{\tilde{z}\tilde{q}}L^{yur}_{\tilde{i}\tilde{z}ij}G_{yu,r}\partial_{\tilde{j}}G_{\tilde{q}[a,b]}e^{-2\lambda f_\epsilon}\cdot\eta_\epsilon^2 dV_g +$$
$$\sum_{a,b=0}^{3}\int_{B_{\epsilon 10}}\sum_{\tilde{i},\tilde{j}=1}^{2}\sum_{\tilde{z},\tilde{q}=0}^{2} g^{\tilde{i}\tilde{j}}g^{\tilde{z}\tilde{q}}L^{yu}_{\tilde{i}\tilde{z}ij}d_{yu}\partial_{\tilde{j}}G_{\tilde{q}[a,b]}e^{-2\lambda f_\epsilon}\cdot\eta_\epsilon^2 dV_g.$$

To control the term in the second line of the RHS we apply the commutation relation again to derive:

$$\sum_{a,b=0}^{3}\int_{B_{\epsilon 10}}\sum_{\tilde{i},\tilde{j}=1}^{2}\sum_{\tilde{z},\tilde{q}=0}^{2} g^{\tilde{i}\tilde{j}}g^{\tilde{z}\tilde{q}}T_{\tilde{i}\tilde{z}ab}\partial_{\tilde{j}}G_{\tilde{q}[a,b]}e^{-2\lambda f_\epsilon}\cdot\eta_\epsilon^2 dV_g = \frac{1}{2}\sum_{a,b=0}^{3}\{\int_{B_{\epsilon 10}}\sum_{\tilde{i},\tilde{j}=1}^{2}\sum_{\tilde{z},\tilde{q}=0}^{2} g^{\tilde{i}\tilde{j}}g^{\tilde{z}\tilde{q}}T_{\tilde{i}\tilde{z}ab}T_{\tilde{j}\tilde{q}ab}e^{-2\lambda f_\epsilon}\eta_\epsilon^2 dV_g$$
$$+\int_{B_{\epsilon 10}} F^{yuqwzxv}_{ab}T_{yuqw}G_{zx,v}e^{-2\lambda f_\epsilon}\cdot\eta_\epsilon^2 dV_g + \int_{B_{\epsilon 10}} F^{yuqwzx}_{ab}T_{yuqw}d_{zx}e^{-2\lambda f_\epsilon}\cdot\eta_\epsilon^2 dV_g\} \leq$$
$$C\{\sum_{a,b,c,d=0}^{3}||T_{abcd}\cdot\eta_\epsilon||^2_{L^2_\lambda} + \sum_{a,b,c=0}^{3}||G_{ab,c}\cdot\eta_\epsilon||^2_{L^2_\lambda} + \sum_{a,b=0}^{3}||d_{ab}\cdot\eta_\epsilon||^2_{L^2_\lambda}\}. \tag{2.53}$$

Now to control the last two lines in the RHS of (2.52) we apply Cauchy-Schwartz to derive that for any $\rho > 0$:

$$\sum_{a,b=0}^{3}\int_{B_{\epsilon 10}}\sum_{\tilde{i},\tilde{j}=1}^{2}\sum_{\tilde{z},\tilde{q}=0}^{2} g^{\tilde{i}\tilde{j}}g^{\tilde{z}\tilde{q}}L^{yur}_{\tilde{i}\tilde{z}ij}G_{yu,r}\partial_{\tilde{j}}G_{\tilde{q}[a,b]}e^{-2\lambda f_\epsilon}\cdot\eta_\epsilon^2 dV_g +$$
$$\sum_{a,b=0}^{3}\int_{B_{\epsilon 10}}\sum_{\tilde{i},\tilde{j}=1}^{2}\sum_{\tilde{z},\tilde{q}=0}^{2} g^{\tilde{i}\tilde{j}}g^{\tilde{z}\tilde{q}}L^{yu}_{\tilde{i}\tilde{z}ij}d_{yu}\partial_{\tilde{j}}G_{\tilde{q}[a,b]}e^{-2\lambda f_\epsilon}\cdot\eta_\epsilon^2 dV_g \leq$$
$$\frac{1}{\rho}\sum_{y,u,r=0}^{3}\int_{B_{\epsilon 10}}(G_{yu,r})^2 e^{-2\lambda f_\epsilon}\cdot\eta_\epsilon^2 dV_g + \frac{1}{\rho}\sum_{y,u=0}^{3}\int_{B_{\epsilon 10}}(d_{yu})^2 e^{-2\lambda f_\epsilon}\cdot\eta_\epsilon^2 dV_g + \tag{2.54}$$
$$C\cdot\rho\cdot\sum_{a,b=0}^{3}\int_{B_{\epsilon 10}}\sum_{\tilde{i},\tilde{j}=1}^{2}\sum_{\tilde{z},\tilde{q}=0}^{2} g^{\tilde{i}\tilde{j}}g^{\tilde{z}\tilde{q}}\partial_{\tilde{i}}G_{\tilde{z}[a,b]}\partial_{\tilde{j}}G_{\tilde{q}[a,b]}e^{-2\lambda f_\epsilon}\cdot\eta_\epsilon^2 dV_g,$$

where the constant C is universal (meaning that it does not depend on λ or ρ– it merely depends on the norms of the tensors g^{ab} etc). We then replace the two equations above into (2.52). Picking ρ small enough we can absorb the term $\sum_{\tilde{z},\tilde{q}=0}^{2} g^{\tilde{i}\tilde{j}}g^{\tilde{z}\tilde{q}}\partial_{\tilde{i}}G_{\tilde{z}[a,b]}\partial_{\tilde{q}}G_{\tilde{q}[a,b]}e^{-2\lambda f_\epsilon}\cdot\eta_\epsilon^2 dV_g$ in the above into the LHS of (2.52). Thus, matters are reduced to controlling the term $\sum_{a,b=0}^{3}\int_{B_{\epsilon 10}}\sum_{\tilde{i},\tilde{j}=1}^{2}\sum_{\tilde{z},\tilde{q}=0}^{2} g^{\tilde{i}\tilde{j}}g^{\tilde{z}\tilde{q}}\partial_{\tilde{z}}G_{\tilde{i}[a,b]}\partial_{\tilde{j}}G_{\tilde{q}[a,b]}e^{-2\lambda f_\epsilon}\cdot\eta_\epsilon^2 dV_g$ in the RHS of (2.52). This can be done by integrating by parts twice, first the derivative $\partial_{\tilde{z}}$ and then the derivative $\partial_{\tilde{j}}$:[14]

$$\sum_{a,b=0}^{3}\int_{B_{\epsilon^{10}}}\sum_{\tilde{i},\tilde{j}=1}^{2}\sum_{\tilde{z},\tilde{q}=0}^{2} g^{\tilde{i}\tilde{j}}g^{\tilde{z}\tilde{q}}\partial_{\tilde{z}}G_{\tilde{i}[a,b]}\partial_{\tilde{j}}G_{\tilde{q}[a,b]}e^{-2\lambda f_\epsilon}\cdot\eta_\epsilon^2 dV_g=\int_{B_{\epsilon^{10}}}V_\epsilon e^{-2\lambda f_\epsilon}dV_g+$$
$$\sum_{a,b=0}^{3}\int_{B_{\epsilon^{10}}}\sum_{\tilde{i},\tilde{j}=1}^{2}\sum_{\tilde{z},\tilde{q}=0}^{2} g^{\tilde{i}\tilde{j}}g^{\tilde{z}\tilde{q}}\partial_{\tilde{j}}G_{\tilde{i}[a,b]}\partial_{\tilde{z}}G_{\tilde{q}[a,b]}e^{-2\lambda f_\epsilon}\cdot\eta_\epsilon^2 dV_g+$$
$$\sum_{a,b=0}^{3}\int_{B_{\epsilon^{10}}} g^{\tilde{i}\tilde{j}}L^{yur}_{ab}G_{yu,r}\partial_{\tilde{i}}G_{\tilde{j}a,b}e^{-2\lambda f_\epsilon}\cdot\eta_\epsilon^2 dV_g+\sum_{a,b=0}^{3}\int_{B_{\epsilon^{10}}}L^{yur}_{ab}G_{yu,r}L'^{yur}_{ab}G_{yu,r}e^{-2\lambda f_\epsilon}\cdot\eta_\epsilon^2 dV_g. \tag{2.55}$$

Applying Cauchy-Schwartz to the expressions $\sum_{a,b=0}^{3}\int_{B_{\epsilon^{10}}} g^{\tilde{i}\tilde{j}}L^{yur}_{ab}G_{yu,r}\partial_{\tilde{i}}G_{\tilde{j}a,b}e^{-2\lambda f_\epsilon}\cdot\eta_\epsilon^2 dV_g$ and $\sum_{a,b=0}^{3}\int_{B_{\epsilon^{10}}} g^{\tilde{i}\tilde{j}}L^{yur}_{ab}G_{yu,r}\partial_{\tilde{i}}G_{\tilde{j}a,b}e^{-2\lambda f_\epsilon}\cdot\eta_\epsilon^2 dV_g$ and then replacing (2.55), (2.53), (2.54) into (2.52) we derive:

$$\sum_{a,b=0}^{3}\int_{B_{\epsilon^{10}}}\sum_{\tilde{i},\tilde{j},\tilde{z},\tilde{q}=\tilde{1}}^{\tilde{2}} g^{\tilde{i}\tilde{j}}g^{\tilde{z}\tilde{q}}\partial_{\tilde{i}}G_{\tilde{z}[a,b]}\partial_{\tilde{j}}G_{\tilde{q}[a,b]}e^{-2\lambda f_\epsilon}\cdot\eta_\epsilon^2 dV_g\le$$
$$\sum_{a,b=0}^{3}\int_{B_{\epsilon^{10}}}|\sum_{\tilde{i},\tilde{j},\tilde{z},\tilde{q}=1}^{2} g^{\tilde{i}\tilde{j}}g^{\tilde{z}\tilde{q}}\partial_{\tilde{i}}G_{\tilde{j}[a,b]}\partial_{\tilde{z}}G_{\tilde{q}[a,b]}e^{-2\lambda f_\epsilon}\cdot\eta_\epsilon^2 dV_g+\sum_{a,b,c,d=0}^{3}||T_{abcd}\cdot\eta_\epsilon||_{L^2_\lambda}+$$
$$\sum_{a,b,c=0}^{3}||\partial_c d_{ab}||_{L^2_\lambda}+\sum_{a,b,c=0}^{3}||\partial^{(2)}_{c0}d_{ab}||_{L^2_\lambda}. \tag{2.56}$$

Finally we again use the fact that $g^{\tilde{3}\tilde{a}}=0$ for ${}^{\tilde{a}}={}^{1},{}^{2},{}^{3}$ to derive:

$$\sum_{a,b=0}^{3}\int_{B_{\epsilon^{10}}}\sum_{\tilde{i},\tilde{j},\tilde{z},\tilde{q}=\tilde{1}}^{\tilde{2}} g^{\tilde{i}\tilde{j}}g^{\tilde{z}\tilde{q}}\partial_{\tilde{i}}G_{\tilde{j}[a,b]}\partial_{\tilde{z}}G_{\tilde{q}[a,b]}\cdot e^{-2\lambda f_\epsilon}\cdot\eta_\epsilon^2 dV_g\le\sum_{a,b,c=0}^{3}||\partial_c d_{ab}||^2_{L^2_\lambda}+$$
$$\sum_{a,b=0}^{3}\int_{B_{\epsilon^{10}}}\sum_{\tilde{i},\tilde{j},\tilde{z},\tilde{q}=\tilde{0}}^{\tilde{3}} g^{\tilde{i}\tilde{j}}g^{\tilde{z}\tilde{q}}\partial_{\tilde{i}}G_{\tilde{j}[a,b]}\partial_{\tilde{z}}G_{\tilde{q}[a,b]}\cdot e^{-2\lambda f_\epsilon}\cdot\eta_\epsilon^2 dV_g+\sum_{a,b,c=0}^{3}||\partial^{(2)}_{c0}d_{ab}||^2_{L^2_\lambda}. \tag{2.57}$$

Since $g^{\tilde{i}\tilde{j}}\partial_{\tilde{i}}G_{\tilde{j}[a,b]}\partial_{\tilde{z}}=D_{[ab]}$, combining this with (2.56) we derive (2.51) and thus our claim. $\Box$

3 A derivation of Theorem 1.1 under minimal hypotheses.

We now show that the conclusion of Theorem 1.1 can in fact be derived under much weaker assumptions. We will see that for two vacuum space-times $(M,\mathbf{g}),(\tilde{M},\tilde{\mathbf{g}})$ with horizons $\mathcal{H}^+\bigcup\mathcal{H}^-$, $\tilde{\mathcal{H}}^+\bigcup\tilde{\mathcal{H}}^-$ to be isometric in some open neighborhoods of $P\in S,\tilde{P}\in\tilde{S}$, it suffices to assume that (near $P,\tilde{P}$) the conformal structures induced

[14] Since the vector fields $\partial_{\tilde{e}},{}_{\tilde{e}}={}_{\tilde{1}},{}_{\tilde{2}}$ are tangent to the level sets of f_ϵ, we do not bring out derivatives of the factor $e^{-\lambda f_\epsilon}$.

by $\mathbf{g}, \tilde{\mathbf{g}}$ onto $(\mathcal{H}^+ \bigcup \mathcal{H}^-)$, $(\tilde{\mathcal{H}}^+ \bigcup \tilde{\mathcal{H}}^-)$ are equivalent,[15] that the spheres $S, \tilde{S}$ with their induced metrics from $\mathbf{g}, \tilde{\mathbf{g}}$ are isometric and moreover that the second fundamental forms of the spheres $S, \tilde{S}$ also agree. The proof of this claim will rely only on an analysis of the Taylor series expansion of the metrics $\mathbf{g}, \tilde{\mathbf{g}}$ on the bifurcate horizons, followed by an application of Theorem 1.1. It is worth noting that this analysis of *free data* is in complete agreement with the work of Rendall in [4]. Nonetheless we are not able to invoke his work directly, since [4] considers the metric expressed in wave coordinates, as opposed to our double Fermi coordinates.

In order to introduce the weaker requirements needed for our stronger version of Theorem 1.1 we define:

Definition 3.1. *We say that the two space-times $(M, \mathbf{g}), (\tilde{M}, \tilde{\mathbf{g}})$ with horizons $\mathcal{H}^+ \bigcup \mathcal{H}^-$, $\tilde{\mathcal{H}}^+ \bigcup \tilde{\mathcal{H}}^-$ are weakly equivalent near points $P \in S = \mathcal{H}^+ \bigcap \mathcal{H}^-$, $\tilde{P} \in \tilde{S} = \tilde{\mathcal{H}}^+ \bigcap \tilde{\mathcal{H}}^-$ if there exist two double Fermi coordinate systems $\mathcal{Y}, \tilde{\mathcal{Y}}$ for M_1, M_2 in open neighborhoods $\Omega, \tilde{\Omega}$ of the points $P, \tilde{P}$ so that the map $\Phi = \tilde{\mathcal{Y}} \circ \mathcal{Y}^{-1}$ satisfies the following properties:*

1. *There exists a function $f \in C^4(\mathcal{H}^+ \bigcup \mathcal{H}^-)$ so that $\Phi^*(\tilde{\mathbf{g}}|_{\tilde{\mathcal{H}}^+ \bigcup \tilde{\mathcal{H}}^- \bigcap \tilde{\Omega}}) = e^{2f}\mathbf{g}|_{\mathcal{H}^+ \bigcup \mathcal{H}^- \bigcap \Omega}$.*
2. *On $S \bigcap \Omega$ we have $f = \frac{\partial}{\partial x^0} f = \frac{\partial}{\partial x^3} f = 0$.*
3. *On $S \bigcap \Omega$ we also have $k_{13} = k'_{13}$, $k_{23} = k'_{23}$, where $\mathbf{k}$, $\mathbf{k}'$ stand for the second fundamental forms of $\mathcal{H}^+$ for the metrics $\mathbf{g}, \Phi^*\mathbf{g}$ with respect to the null vector field $\frac{\partial}{\partial x^3}$.*

Our strengthened theorem is then the following:

Theorem 3.1. *Consider two vacuum space-times $(M, \mathbf{g}), (\tilde{M}, \tilde{\mathbf{g}})$ as in the discussion above Theorem 1.1, with horizons $(\mathcal{H}^+ \bigcup \mathcal{H}^-)$, $(\tilde{\mathcal{H}}^+ \bigcup \tilde{\mathcal{H}}^-)$.*

Then if these two horizons are weakly equivalent near the points $P, \tilde{P}$, then the map Φ in the definition above is an isometry, when restricted to a small enough open neighborhood of $P, \tilde{P}$.

Proof of Theorem 3.1: We only need to show that $\Phi^*\tilde{\mathbf{g}} - \mathbf{g}$ vanishes to third order on $(\mathcal{H}^+ \bigcup \mathcal{H}^-) \bigcap \Omega$; Theorem 1.1 will then imply that $(M, \mathbf{g}), (\tilde{M}, \tilde{\mathbf{g}})$ are isometric in open neighborhoods of $P, \tilde{P}$. We first prove this claim on $\mathcal{H}^+$.

Proof of the claim on $\mathcal{H}^+$: We will show this in two steps: Firstly we will prove that $f = 0$ on $\mathcal{H}^+$. Then we will show that the jets up to third order of $\mathbf{g}$, $\Phi^*\tilde{\mathbf{g}}$ are uniquely determined given the values of k_{13}, k_{23}, k'_{13}, k'_{23}. Given the hypothesis of our Lemma, that will prove that $\Phi^*\tilde{\mathbf{g}} - \mathbf{g}$ vanishes to third order on $\mathcal{H}^+$. We introduce a notational convention to simplify our task: At each stage of our proof we will denote by $K(x^1, x^2, x^3), K'(x^1, x^2, x^3), \dots$ (or just $K, K', \dots$ for short) generic *known* functions defined over $\mathcal{H}^+$.

We first show that $f = 0$ on $\mathcal{H}^+$: Consider the equation $Ric_{33}(\Phi^*\tilde{\mathbf{g}}) - Ric_{33}(\mathbf{g}) = 0$ on $\mathcal{H}^+$. We thus derive an equation:

$$\partial^{(2)}_{33} f - (\partial_3 f)^2 = K(x^1, x^2, x^3)\partial_3 f$$

Thus, given that $f = \partial_0 f = 0$ in S, we derive that $f = 0$ by the fundamental theorem of ODEs.

Now, we recall that $g_{00} = \tilde{g}_{00} = g_{01} = \tilde{g}_{01} = g_{02} = \tilde{g}_{02} = 0$ throughout Ω. Furthermore, since the integral curves of $\frac{\partial}{\partial x^3}$ in $\mathcal{H}^+_1$ are null geodesics, and the fact that $g_{33} = g_{13} = g_{23} = 0$ on S, we derive that $g_{33} = g_{13} = g_{23} = 0$ throughout $\mathcal{H}^+$. Finally, since integral curves are arc-length paramatrized geodesics, we derive that $\partial_0 g_{33} = 0$ throughout $\mathcal{H}^+$. (The same relations are of course true for the corresponding components of $(\Phi^*\tilde{g})$).

[15] This notion will be made precise below.

Now we will show that the components $\partial^{(k)}_{0\dots0}g_{13}, \partial^{(k)}_{0\dots0}g_{23}, \partial^{(k)}_{0\dots0}g_{11}, \partial^{(k)}_{0\dots0}g_{12}, \partial^{(k)}_{0\dots0}g_{22}$ for $1 \le k \le 2$, and $\partial^{(2)}_{00}g_{33}$ on $\mathcal{H}^+$ can be uniquely determined by the above relations given the equation $Ric(g) = 0$ and also the data $\partial_0 g_{13}$, $\partial_0 g_{23}$ on S. This will *also* show that the corresponding components of $\Phi^*\tilde{g}, \partial(\Phi^*\tilde{g}), \partial^{(2)}(\Phi^*\tilde{g})$ are uniquely determined by the equation $Ric(\Phi^*\tilde{g}) = 0$ and the $\partial_0(\Phi^*\tilde{g})_{13}$, $\partial_0(\Phi^*\tilde{g})_{23}$ on S and we will thus derive our claim on $\mathcal{H}^+$.

We determine the above components of the jet of g in stages: Firstly observe that in the notation of Definition 3.1, for any point on S: $k_{13} = \frac{1}{2}\partial_0 g_{13}$, $k'_{13} = \frac{1}{2}\partial_0(\Phi^*\tilde{g}_{13})$ and also $k_{23} = \frac{1}{2}\partial_0 g_{23}$, $k'_{23} = \frac{1}{2}\partial_0(\Phi^*\tilde{g}_{23})$. Next, we will derive ODEs involving the unknowns $\partial_0 g_{13}, \partial_0 g_{23}$ in $\mathcal{H}^+$: Consider the equations $Ric_{13} = 0, Ric_{23} = 0$. We derive:

$$\partial_3(\partial_0 g_{13}) = K(x^1, x^2, x^3), \partial_3(\partial_0 g_{23}) = K(x^1, x^2, x^3).$$

Thus, again by the fundamental theorem of ODEs we determine the functions $\partial_0 g_{13}, \partial_0 g_{23}$ on $\mathcal{H}^+$. Now we use the equations $Ric_{11} = 0, Ric_{12} = 0, Ric_{22} = 0$ to derive:

$$\partial_3(\partial_0 g_{11}) = K\partial_0 g_{11} + K'\partial_0 g_{12} + K''\partial_0 g_{22} + K''', \tag{3.58}$$
$$\partial_3(\partial_0 g_{12}) = K(x^1, x^2, x^3)\partial_0 g_{11} + K'\partial_0 g_{12} + K''\partial_0 g_{22} + K''', \tag{3.59}$$
$$\partial_3(\partial_0 g_{22}) = K\partial_0 g_{11} + K'\partial_0 g_{12} + K''\partial_0 g_{22} + K'''. \tag{3.60}$$

Thus we again invoke the fundamental theorem of ODEs to conclude that the functions $\partial_0 g_{11}, \partial_0 g_{12}, \partial_0 g_{22}$ are uniquely determined on $\mathcal{H}^+$ by the data we have prescribed.

Similarly, considering the equation $Ric_{03} = 0$ on $\mathcal{H}^+$ we calculate $\partial^{(2)}_{00}g_{33}$ and then using the equations $Ric_{01} = Ric_{02} = 0$ we calculate $\partial^{(2)}_{00}g_{13}, \partial^{(2)}_{00}g_{13}$ on $\mathcal{H}^+$. Finally, considering the equations $\partial_0 Ric_{11} = 0, \partial_0 Ric_{12} = 0, \partial_0 Ric_{22} = 0$ and repeating the argument we used for the equations (3.58), (3.58), (3.59) we calculate $\partial^{(2)}_{00}g_{11}, \partial^{(2)}_{00}g_{12}, \partial^{(2)}_{00}g_{22}$. This proves our claim on $\mathcal{H}^+$.

Proof of the claim on $\mathcal{H}^-$: Again, we only need to show that $f = 0$ on $\mathcal{H}^-$ and that the 2-jets of g are uniquely determined by the equation $Ric(g) = 0$ and the data $\partial_0(g_{13})$, $\partial_0(g_{23})$ on S.

Again, at each stage of our proof we will denote by $K(x^0, x^1, x^2), K'(x^0, x^1, x^2), \dots$ (or just $K, K', \dots$ for short) generic *known* functions defined over $\mathcal{H}^-$.

To show that $f = 0$ on $\mathcal{H}^-$ we consider the equation $Ric_{00}(\Phi^*\tilde{g}) - Ric_{00}(g) = 0$ on $\mathcal{H}^-$. We derive an ODE on the conformal factor f:

$$\partial^{(2)}_{00}f + (Const)\cdot(\partial_0 f)^2 + K\partial_0 f = 0$$

Thus we derive that $f = 0$ on $\mathcal{H}^-$ using the fundamental theorem of ODE and the hypothesis that $f = \partial_3 f = 0$ on S.

Now, the unknowns we wish to determine are the components $\partial^{(k)}_{3\dots3}g_{13}, \partial^{(k)}_{3\dots3}g_{23}, \partial^{(k)}_{3\dots3}g_{33}$ for every $k, 0 \le k \le 2$ and the components $\partial^{(k)}_{3\dots3}g_{11}$, $\partial^{(k)}_{3\dots3}g_{12}$, $\partial^{(k)}_{3\dots3}g_{22}$ for every $k, 1 \le k \le 2$.

Using the equations $Ric_{01} = 0$, $Ric_{02} = 0$ we derive:

$$\partial^{(2)}_{00}g_{13} + K\partial_0 g_{13} + K'g_{23} = K'', \partial^{(2)}_{00}g_{23} + \tilde{K}\partial_0 g_{13} + \tilde{K}'g_{23} = \tilde{K}''.$$

Therefore by applying the fundamental theorem of ODEs to the above two equations we derive that we can determine g_{13}, g_{23} from the initial data on S. Now we also consider the equations $Ric_{11} = 0, Ric_{12} = 0, Ric_{22} = 0$. We derive a system of three equations:

$$\partial_0(\partial_3 g_{11}) + K_1\partial_3 g_{11} + K_2\partial_3 g_{12} + K_3\partial_3 g_{22} = K_4, \tag{3.61}$$
$$\partial_0(\partial_3 g_{12}) + \tilde{K}_1\partial_3 g_{11} + \tilde{K}_2\partial_3 g_{12} + \tilde{K}_3\partial_3 g_{22} = \tilde{K}_4, \tag{3.62}$$
$$\partial_0(\partial_3 g_{22}) + \overline{K}_1\partial_3 g_{11} + \overline{K}_2\partial_3 g_{12} + \overline{K}_3\partial_3 g_{22} = \overline{K}_4. \tag{3.63}$$

Thus, we also derive that the values of $\partial_0 g_{11}, \partial_0 g_{12}, \partial_0 g_{22}$ can be determined on $\mathcal{H}^-$ from our prescribed data.

Finally, using the equation $Ric_{03} = 0$ on $\mathcal{H}^-$ we derive an equation:

$$\partial^{(2)}_{00} g_{33} + K_1\partial_0 g_{33} = K_2.$$

Therefore, using the fact that $\partial_0 g_{33} = 0$ on S we derive that g_{33} can also be determined on $\mathcal{H}^-$. Finally considering ∂_3-derivatives of the equations above, we can also determine all the higher derivatives of the unknown functions. □

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