HAWKING'S LOCAL RIGIDITY THEOREM WITHOUT ANALYTICITY

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Abstract. We prove the existence of a Hawking Killing vector-field in a full neighborhood of a local, regular, bifurcate, non-expanding horizon embedded in a smooth vacuum Einstein manifold. The result extends a previous result of Friedrich, Rácz and Wald, see [7, Proposition B.1], which was limited to the domain of dependence of the bifurcate horizon. So far, the existence of a Killing vector-field in a full neighborhood has been proved only under the restrictive assumption of analyticity of the space-time. We also prove that, if the space-time possesses an additional Killing vectorfield $\mathbf{T}$, tangent to the horizon and not vanishing identically on the bifurcation sphere, then there must exist a local rotational Killing field commuting with $\mathbf{T}$. Thus the space-time must be locally axially symmetric. The existence of a Hawking vector-field $\mathbf{K}$, and the above mentioned axial symmetry, plays a fundamental role in the classification theory of stationary black holes. In [2] we use the results of this paper to prove a perturbative version of the uniqueness of smooth, stationary black holes in vacuum.

Contents

1. Introduction

Let $(\mathbf{M}, \mathbf{g})$ to be a smooth[1] vacuum Einstein space-time. Let S be an embedded spacelike 2-sphere in $\mathbf{M}$ and let $\mathcal{N}, \underline{\mathcal{N}}$ be the null boundaries of the causal set of S, i.e. the union of the causal future and past of S. We fix $\mathbf{O}$ to be a small neighborhood of S such that both $\mathcal{N}, \underline{\mathcal{N}}$ are regular, achronal, null hypersurfaces in $\mathbf{O}$ spanned by null geodesic generators orthogonal to S. We say that the triplet $(S, \mathcal{N}, \underline{\mathcal{N}})$ forms a

[1] $\mathbf{M}$ is assumed to be a connected, oriented, C^∞ 4-dimensional manifold without boundary.

local, regular, bifurcate, non-expanding horizon in $\mathcal{O}$ if both $\mathcal{N}, \underline{\mathcal{N}}$ are non-expanding null hypersurfaces (see definition 2.1) in $\mathcal{O}$. Our main result is the following:

Theorem 1.1. *Given a local, regular, bifurcate, non-expanding horizon* $(S, \mathcal{N}, \underline{\mathcal{N}})$ *in a smooth, vacuum Einstein space-time* $(\mathbf{O}, \mathbf{g})$*, there exists an open neighborhood* $\mathbf{O}' \subseteq \mathbf{O}$ *of* S *and a non-trivial Killing vector-field* $\mathbf{K}$ *in* $\mathbf{O}'$*, which is tangent to the null generators of* $\mathcal{N}$ *and* $\underline{\mathcal{N}}$*. In other words, every local, regular, bifurcate, non-expanding horizon is a Killing bifurcate horizon.*

It is already known, see [7], that such a Killing vector-field exists in a small neighborhood of S intersected with the domain of dependence of $\mathcal{N} \cup \underline{\mathcal{N}}$. The extension of $\mathbf{K}$ to a full neighborhood of S has been known to hold only under the restrictive additional assumption of analyticity of the space-time (see [8], [12], [7]). The novelty of our theorem is the existence of Hawking's Killing vector-field $\mathbf{K}$ in a full neighborhood of the 2-sphere S, without making any analyticity assumption. It is precisely this information, i.e. the existence of $\mathbf{K}$ in the complement of the domain of dependence of $\mathcal{N} \cup \underline{\mathcal{N}}$, that is needed in the application of Hawking's rigidity theorem to the classification theory of stationary, regular black holes. The assumption that the non-expanding horizon in Theorem 1.1 is *bifurcate* is essential for the proof; this assumption is consistent with the application mentioned above.

We also prove the following:

Theorem 1.2. *Assume that* $(S, \mathcal{N}, \underline{\mathcal{N}})$ *is a local, regular, bifurcate, horizon in a vacuum Einstein space-time* $(\mathbf{O}, \mathbf{g})$ *which possesses a Killing vectorfield* $\mathbf{T}$ *tangent to* $\mathcal{N} \cup \underline{\mathcal{N}}$ *and non-vanishing on* S*. Then, there exists an open neighborhood* $\mathbf{O}' \subseteq \mathbf{O}$ *of* S *and a non-trivial rotational Killing vector-field* $\mathbf{Z}$ *in* $\mathbf{O}'$ *which commutes with* $\mathbf{T}$*.*

Once more, a related version of result was known only in the special case when the space-time is analytic. In fact S. Hawking's famous rigidity theorem, see [8], asserts that, under some global causality, asymptotic flatness and connectivity assumptions, a stationary, non-degenerate analytic spacetime must be axially symmetric. Observe that, though we have not assumed specifically that the horizon is non-expanding, this is in fact a well known consequence of the fact that the Killing field $\mathbf{T}$ is tangent to it. Thus, in view of Theorem 1.1, there exists a Hawking vectorfield $\mathbf{K}$, in a full neighborhood of S. We show that there exist constants λ_0 and $t_0 > 0$ such that

$$\mathbf{Z} = \mathbf{T} + \lambda_0 \mathbf{K} \tag{1.1}$$

is a rotation with period t_0. The main constants λ_0 and t_0 can be determined on the bifurcation sphere S. We remark that, though Hawking's rigidity theorem does not require, explicitely, a regular, bifurcate horizon, our assumption is related to that of the non-degeneracy of the event horizon, see [17].

As known the existence of the Hawking vector-field plays a fundamental role in the classification theory of stationary black holes (see [8] or [5] and references therein for a more complete treatment of the problem). The results of this paper are used in [2] to prove a perturbative version, without analyticity, of the uniqueness of smooth, stationary black holes in vacuum. More precisely we show that a regular, smooth, asymptotically flat solution of the vacuum Einstein equations which is a perturbation of a Kerr solution $\mathcal{K}(a,m)$ with $0 \le a < m$ is in fact a Kerr solution. The perturbation condition is expressed geometrically by assuming that the Mars-Simon tensor SS of the stationary space-time (see [15] and [10]) is sufficiently small. The proof uses Theorem 1.1 as a first step; one first defines a Hawking vector-field $\mathbf{K}$ in a neighborhood of S and then extends it to the entire space-time by using the level sets of a canonically defined function y. One can show that these level sets are conditionally pseudo-convex, as in [10], as long as the the Mars-Simon tensor SS is sufficiently small. Once $\mathbf{K}$ is extended to the entire space-time one can show, using the result of Theorem 1.2, that the space-time is not only stationary but also axisymmetric. The proof then follows by appealing to the methods of the well known results of Carter [3] and Robinson [16], see also the more complete account [5].

1.1. Main Ideas.

We recall that a Killing vector-field $\mathbf{K}$ in a vacuum Einstein space-time must verify the covariant wave equation

$$\square_{\mathbf{g}}\mathbf{K} = 0. \tag{1.2}$$

The main idea in [7] was to construct $\mathbf{K}$ as a solution to (1.2) with appropriate, characteristic, boundary conditions on $\mathcal{N} \cup \underline{\mathcal{N}}$. As known, the characteristic initial value problem is well posed in the domain of dependence of $\mathcal{N} \cup \underline{\mathcal{N}}$ but ill posed in its complement. To avoid this fundamental difficulty we rely instead on a completely different strategy[2]. The main idea, which allows us to avoid using (1.2) or some other system of PDE's in the ill posed region, is to first construct $\mathbf{K}$ in the domain of dependence of $\mathcal{N} \cup \underline{\mathcal{N}}$ as a solution to (1.2), extend $\mathbf{K}$ by Lie dragging along the null geodesics transversal to $\mathcal{N}$, consider its associated flow Ψ_t, and show that, for small $|t|$, the pull back metric $\Psi_t^*\mathbf{g}$ must coincide with $\mathbf{g}$, in view of the fact they they are both solutions of the Einstein vacuum equations and coincide on $\mathcal{N} \cup \underline{\mathcal{N}}$. To implement this idea we need to prove a uniqueness result for two Einstein vacuum metrics $\mathbf{g}$, $\mathbf{g}'$ which coincide on $\mathcal{N} \cup \underline{\mathcal{N}}$. Such a uniqueness result was proved by one of the authors in [1], based on the uniqueness results for systems of covariant wave equations proved by the other two authors in [10] and [11]. The starting point of the proof are the schematic identities,

$$\square_{\mathbf{g}}\mathbf{R} = \mathbf{R} * \mathbf{R}, \qquad \square_{\mathbf{g}'}\mathbf{R}' = \mathbf{R}' * \mathbf{R}'$$

with $\mathbf{R} * \mathbf{R}$, $\mathbf{R}' * \mathbf{R}'$ quadratic expressions in the curvatures $\mathbf{R}, \mathbf{R}'$ of the Einstein vacuum metrics $\mathbf{g}, \mathbf{g}'$. Subtracting the two equations we derive,

$$\square_{\mathbf{g}}(\mathbf{R} - \mathbf{R}') + \big(\square_{\mathbf{g}} - \square_{\mathbf{g}'}\big)\mathbf{R}' = (\mathbf{R} - \mathbf{R}') * (\mathbf{R} + \mathbf{R}').$$

[2] Such a strategy was discussed in [7, Remark B.1.], as an alternative to the use of the wave equation (1.2), in the domain of dependence of $\mathcal{N} \cup \underline{\mathcal{N}}$. We would like to thank R. Wald for drawing our attention to it.

We would like to rely on the uniqueness properties of covariant wave equations, as in [10], [11], but this is not possible due to the presence of the term $(\square_{\mathbf{g}} - \square_{\mathbf{g}'})\mathbf{R}'$ which forces us to consider equations for $\mathbf{g} - \mathbf{g}'$ expressed relative to an appropriate choice of a gauge condition. An obvious such gauge choice would be the wave gauge $\square_{\mathbf{g}} x^\alpha = 0$ which would lead to a system of wave equations for the components of the two metrics $\mathbf{g}$, $\mathbf{g}'$ in the given coordinate system. Unfortunately such coordinate system would have to be constructed starting with data on $\mathcal{N} \cup \underline{\mathcal{N}}$ which requires one to solve the same ill posed problem. We rely instead on a pair of geometrically constructed frames v, v' (using parallel transport with respect to $\mathbf{g}$ and $\mathbf{g}'$) and derive ODE's for their difference $dv = v' - v$, as well as the difference $d\Gamma = \Gamma' - \Gamma$ between their connection coefficients, with source terms in $dR = \mathbf{R}' - \mathbf{R}$. In this way we derive a system of wave equations in dR coupled with ODE's in dv, $d\Gamma$ and their partial derivatives ∂dv, $\partial d\Gamma$ with respect to our fixed coordinate system. Since ODE's are clearly well posed it is natural to expect that the uniqueness results for covariant wave equations derived in [10], [11] can be extended to such coupled system and thus deduce that $dv = d\Gamma = dR = 0$ in a full neighborhood of S. The precise result is stated and proved in Lemma 4.4.

In section 2 we construct a canonical null frame which will be used throughout the paper. We use the non-expanding condition to derive the main null structure equations along $\mathcal{N}$ and $\underline{\mathcal{N}}$. In section 3 we give a self contained proof of Proposition B.1. in [7] concerning the existence of a Hawking vector-field in the domain of dependence of $\mathcal{N} \cup \underline{\mathcal{N}}$. In section 4, we show how to extend $\mathbf{K}$ to a full neighborhood of S. We also show that the extension must be locally time-like in the complement of the domain of dependence of $\mathcal{N} \cup \underline{\mathcal{N}}$, see Proposition 4.5. In section 5 we prove Theorem 1.2. We first show that if $\mathbf{T}$ is another smooth Killing vector-field, tangent to $\mathcal{N} \cup \underline{\mathcal{N}}$, then it must commute with $\mathbf{K}$ in a full neighborhood of S. We then construct a rotational Killing vector-field $\mathbf{Z}$ as a linear combination of $\mathbf{T}$ and $\mathbf{K}$. We also show that if σ_μ is the Ernst potential associated with $\mathbf{T}$ then $\mathbf{K}^\mu = \mathbf{Z}^\mu \sigma_\mu = 0$. These additional results, in the presence of the (stationary) Killing vector-field $\mathbf{T}$, are important in the application in [2].

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2. Preliminaries

We restrict our attention to an open neighborhood $\emptyset$ of S in which $\mathcal{N}, \underline{\mathcal{N}}$ are regular, achronal, null hypersurfaces, spanned by null geodesic generators orthogonal to S. During the proof of our main theorem and their consequences we will keep restricting our attention to smaller and smaller neighborhoods of S; for simplicity of notation we keep denoting such neighborhoods of S by $\emptyset$.

We define two optical functions $u, \underline{u}$ in a neighborhood of S as follows. We first fix a smooth future-directed null pair $(L, \underline{L})$ along S, satisfying

$$\mathbf{g}(L, L) = \mathbf{g}(\underline{L}, \underline{L}) = 0, \quad \mathbf{g}(L, \underline{L}) = -1, \tag{2.1}$$

such that L is tangent to $\mathcal{N}$ and $\underline{L}$ is tangent to $\underline{\mathcal{N}}$. In a small neighborhood of S, we extend L (resp. $\underline{L}$) along the null geodesic generators of $\mathcal{N}$ (resp. $\underline{\mathcal{N}}$) by parallel transport, i.e. $\mathbf{D}_L L = 0$ (resp. $\mathbf{D}_{\underline{L}} \underline{L} = 0$). We define the function $\underline{u}$ (resp. u) along $\mathcal{N}$ (resp. $\underline{\mathcal{N}}$) by setting $u = \underline{u} = 0$ on S and solving $L(\underline{u}) = 1$ (resp. $\underline{L}(u) = 1$). Let $S_{\underline{u}}$ (resp. $\underline{S}_u$) be the level surfaces of $\underline{u}$ (resp. u) along $\mathcal{N}$ (resp. $\underline{\mathcal{N}}$). We define $\underline{L}$ at every point of $\mathcal{N}$ (resp. L at every point of $\underline{\mathcal{N}}$) as the unique, future directed null vector-field orthogonal to the surface $S_{\underline{u}}$ (resp. $\underline{S}_u$) passing through that point and such that $\mathbf{g}(L, \underline{L}) = -1$. We now define the null hypersurface $\underline{\mathcal{N}}_{\underline{u}}$ to be the congruence of null geodesics initiating on $S_{\underline{u}} \subset \mathcal{N}$ in the direction of $\underline{L}$. Similarly we define $\mathcal{N}_u$ to be the congruence of null geodesics initiating on $\underline{S}_u \subset \underline{\mathcal{N}}$ in the direction of L. Both congruences are well defined in a sufficiently small neighborhood of S in $\emptyset$, which (according to our convention) we continue to call $\emptyset$. The null hypersurfaces $\underline{\mathcal{N}}_{\underline{u}}$ (resp. $\mathcal{N}_u$) are the level sets of a function $\underline{u}$ (resp u) vanishing on $\underline{\mathcal{N}}$ (resp. $\mathcal{N}$). By construction

$$L = -\mathbf{g}^{\mu\nu} \partial_\mu u \partial_\nu, \qquad \underline{L} = -\mathbf{g}^{\mu\nu} \partial_\mu \underline{u} \partial_\nu. \tag{2.2}$$

In particular, the functions $u, \underline{u}$ are both null optical functions, i.e.

$$\mathbf{g}^{\mu\nu} \partial_\mu u \partial_\nu u = \mathbf{g}(L, L) = 0 \quad \text{ and } \quad \mathbf{g}^{\mu\nu} \partial_\mu \underline{u}\, \partial_\nu \underline{u} = \mathbf{g}(\underline{L}, \underline{L}) = 0. \tag{2.3}$$

We define,

$$\Omega = \mathbf{g}^{\mu\nu} \partial_\mu u\, \partial_\nu \underline{u} = \mathbf{g}(L, \underline{L}).$$

By construction $\Omega = -1$ on $(\mathcal{N} \cup \underline{\mathcal{N}}) \cap \emptyset$, but Ω is not necessarily equal to -1 in $\emptyset$. Choosing $\emptyset$ small enough, we may assume however that $\Omega \in [-3/2, -1/2]$ in $\emptyset$.

To summarize, we can find two smooth optical functions $u, \underline{u} : \emptyset \to \mathbb{R}$ such that,

$$\mathcal{N} \cap \emptyset = \{p \in \emptyset : u(p) = 0\}, \qquad \underline{\mathcal{N}} \cap \emptyset = \{p \in \emptyset : \underline{u}(p) = 0\}. \tag{2.4}$$

and,

$$\Omega \in [-3/2, -1/2] \quad \text{ in } \emptyset. \tag{2.5}$$

Moreover, by construction (with L, $\underline{L}$ defined by (2.2)) we have,

$$L(\underline{u}) = 1 \text{ on } \mathcal{N}, \qquad \underline{L}(u) = 1 \text{ on } \underline{\mathcal{N}}.$$

Using the null pair $\underline{L}, L$ introduced in (2.1), (2.2) we fix an associated null frame $e_1, e_2, e_3 = \underline{L}, e_4 = L$ such that $\mathbf{g}(e_a, e_a) = 1$, $\mathbf{g}(e_1, e_2) = \mathbf{g}(e_4, e_a) = \mathbf{g}(e_3, e_a) = 0$, $a = 1, 2$. At every point p in in $\emptyset$, e_1, e_2 form an orthonormal frame along the 2-surface $S_{u,\underline{u}}$ passing through p. We denote by ∇ the induced covariant derivative operator on $S_{u,\underline{u}}$. Given a horizontal vector-field X, i.e. X tangent to the 2-surfaces $S_{u,\underline{u}}$ at every point in $\emptyset$, we denote by $\nabla_3 X$, $\nabla_4 X$ the projections of $\mathbf{D}_{e_3}$ and $\mathbf{D}_{e_4}$ to $S_{u,\underline{u}}$. Recall the definition of the null second fundamental forms

$$\chi_{ab} = \mathbf{g}(\mathbf{D}_{e_a} L, e_b), \qquad \underline{\chi}_{ab} = \mathbf{g}(\mathbf{D}_{e_a} \underline{L}, e_b)$$

and the torsion

$$\zeta_a = \mathbf{g}(\mathbf{D}_{e_a} L, \underline{L}).$$

Definition 2.1. *We say that $\mathcal{N}$ is non-expanding if $tr\,\chi = 0$ on $\mathcal{N}$. Similarly $\underline{\mathcal{N}}$ is non-expanding if $tr\,\underline{\chi} = 0$ on $\underline{\mathcal{N}}$. The bifurcate horizon $(S, \mathcal{N}, \underline{\mathcal{N}})$ is called non-expanding if both $\mathcal{N}, \underline{\mathcal{N}}$ are non-expanding.*

The assumption that the surfaces $\mathcal{N}$ and $\underline{\mathcal{N}}$ are non-expanding implies, according to the Raychadhouri equation,

$$\chi = 0 \text{ on } \mathcal{N} \cap \emptyset \quad \text{and} \quad \underline{\chi} = 0 \text{ on } \underline{\mathcal{N}} \cap \emptyset. \tag{2.6}$$

In addition, since the vectors e_1, e_2 are tangent to $(\mathcal{N} \cup \underline{\mathcal{N}}) \cap \emptyset$ and $\mathbf{g}(L, \underline{L}) = -1$ on $(\mathcal{N} \cup \underline{\mathcal{N}}) \cap \emptyset$, we have $\zeta_a = -\mathbf{g}(D_{e_a} \underline{L}, L)$ on $(\mathcal{N} \cup \underline{\mathcal{N}}) \cap \emptyset$. Finally, it is known that the following components of the curvature tensor $\mathbf{R}$ vanish on $\mathcal{N}$ and $\underline{\mathcal{N}}$,

$$\mathbf{R}_{4a4b} = \mathbf{R}_{434b} = 0 \text{ on } \mathcal{N} \quad \text{and} \quad \mathbf{R}_{3a3b} = \mathbf{R}_{343b} = 0 \text{ on } \underline{\mathcal{N}}, \quad a, b = 1, 2. \tag{2.7}$$

Let, see [4], [14], $\alpha_{ab} = \mathbf{R}_{4a4b}$, $\beta_a = \mathbf{R}_{a434}$, $\rho = \mathbf{R}_{3434}$, $\sigma = {}^*\mathbf{R}_{3434}$, $\underline{\beta}_a = \mathbf{R}_{a334}$ and $\underline{\alpha}_{ab} = \mathbf{R}_{a3b3}$ denote the null components of $\mathbf{R}$. Thus, in view of (2.7) the only non-vanishing null components of $\mathbf{R}$ on S are ρ and σ. Since $[e_a, e_4](\underline{u}) = 0$ on $\mathcal{N} \cap \emptyset$, it follows that $\mathbf{g}([e_a, e_4], e_3) = 0$ on $\mathcal{N} \cap \emptyset$. Using $\mathbf{D}_L L = 0$, (2.6), and the definitions, we derive, on $\mathcal{N} \cap \emptyset$,

$$\begin{aligned} &\mathbf{D}_{e_4} e_4 = 0, \quad \mathbf{D}_{e_a} e_4 = -\zeta_a e_4, \quad \mathbf{D}_{e_4} e_3 = -\sum_{a=1}^{2} \zeta_b e_b, \quad \mathbf{D}_{e_4} e_a = \nabla_{e_4} e_a - \zeta_a e_4, \\ &\mathbf{D}_{e_a} e_3 = \sum_{b=1}^{2} \underline{\chi}_{ab} e_b + \zeta_a e_3, \quad \mathbf{D}_{e_a} e_b = \nabla_{e_a} e_b + \underline{\chi}_{ab} e_4. \end{aligned} \tag{2.8}$$

Lemma 2.2. *The null structure equations along $\mathcal{N}$ (see[3] Proposition 3.1.3 in [14]) reduce to*

$$\nabla_4 \zeta = 0, \quad curl\, \zeta = \sigma, \quad L(tr\, \underline{\chi}) + div\, \zeta - |\zeta|^2 = \rho. \tag{2.9}$$

Also, if X is an horizontal vector,

$$[\nabla_4, \nabla_a] X_b = 0.$$

As a consequence we also have,

$$\nabla_4 (div\, \zeta) = 0. \tag{2.10}$$

Proof of Lemma 2.2. Indeed,

$$\mathbf{g}(\mathbf{D}_4 \mathbf{D}_a \underline{L}, e_4) - \mathbf{g}(\mathbf{D}_a \mathbf{D}_4 \underline{L}, e_4) = \mathbf{R}(e_a, e_4, e_3, e_4) = \beta_a$$

and, using (2.8), $\mathbf{g}(\mathbf{D}_a \mathbf{D}_4 \underline{L}, e_4) = \underline{L}_{4;4a} = 0$, $\mathbf{g}(\mathbf{D}_4 \mathbf{D}_a \underline{L}, e_4) = \underline{L}_{4;a4} = -\nabla_4 \zeta_a$. Hence, since β vanishes along $\mathcal{N}$, we deduce $\nabla_4 \zeta = 0$. Also,

$$\mathbf{g}(\mathbf{D}_4 \mathbf{D}_b \underline{L}, e_a) - \mathbf{g}(\mathbf{D}_b \mathbf{D}_4 \underline{L}, e_a) \;=\; \mathbf{R}(e_a, e_3, e_4, e_b) = \frac{1}{2} \gamma_{ab}\, \rho - \frac{1}{2} \in_{ab} \sigma$$

[3]The discrepancy with the corresponding formula is due to the different normalization for $\underline{L}$, i.e. $\mathbf{g}(L, \underline{L}) = -1$ instead of $\mathbf{g}(L, \underline{L}) = -2$.

and, $\mathbf{g}(\mathbf{D}_4\mathbf{D}_b\,\underline{L}, e_a) = \underline{L}_{a;b4} = \nabla_4\underline{\chi}_{ab} - 2\zeta_a\zeta_b$, $g(\mathbf{D}_b\mathbf{D}_4\,\underline{L}, e_a) = \underline{L}_{a;4b} = -\nabla_b\zeta_a - \zeta_a\zeta_b$. Hence,

$$\nabla_4\underline{\chi}_{ab} - \zeta_a\zeta_b + \partial_b\zeta_a = \frac{1}{2}\rho\,\gamma_{ab} - \frac{1}{2}\sigma \in_{ab} .$$

Taking the symmetric part we derive, $\nabla_4 \mathrm{tr}\,\underline{\chi} - |\zeta|^2 + \mathrm{d}iv\,\zeta = \rho$ while taking the antisymmetric part yields, $curl\,\zeta = \sigma$ as desired. To check the commutation formula we write,

$$\begin{aligned}
\mathbf{D}_4\mathbf{D}_aX_b &= e_4(\mathbf{D}_aX_b) - \mathbf{D}_{\mathbf{D}_4e_a}X_b - \mathbf{D}_aX_{\mathbf{D}_4e_b}\\
&= e_4(\nabla_bX_a) - \mathbf{D}_{\nabla_4e_a}X_b + \zeta_a\mathbf{D}_4X_b - \mathbf{D}_aX_{\nabla_4e_a} + \zeta_b\mathbf{D}_aX_4\\
&= \nabla_4\nabla_aX_b + \zeta_a\nabla_4X_b\\
\mathbf{D}_a\mathbf{D}_4X_b &= e_a(\mathbf{D}_4X_b) - \mathbf{D}_{\mathbf{D}_ae_4}X_b - \mathbf{D}_4X_{\mathbf{D}_ae_b}\\
&= e_a(\mathbf{D}_4X_b) - \mathbf{D}_{\nabla_ae_4}X_b + \zeta_a\mathbf{D}_4X_b - \mathbf{D}_4X_{\nabla_ae_b}\\
&= \nabla_a\nabla_4X_b + \zeta_a\nabla_4X_b
\end{aligned}$$

Therefore,

$$[\mathbf{D}_4, \mathbf{D}_a]X_b = [\nabla_4, \nabla_a]X_b.$$

On the other hand, $[\mathbf{D}_4, \mathbf{D}_a]X_b = \mathbf{R}_{a4cb}X^c = 0$ in view of the vanishing of β and the Einstein equations. □

We define the following four regions I^{++}, I^{--}, I^{+-} and I^{-+}:

$$\begin{aligned}
I^{++} &= \{p \in \emptyset : u(p) \geq 0 \text{ and } \underline{u}(p) \geq 0\}, \quad I^{--} = \{p \in \emptyset : u(p) \leq 0 \text{ and } \underline{u}(p) \leq 0\},\\
I^{+-} &= \{p \in \emptyset : u(p) \geq 0 \text{ and } \underline{u}(p) \leq 0\}, \quad I^{-+} = \{p \in \emptyset : u(p) \leq 0 \text{ and } \underline{u}(p) \geq 0\}.
\end{aligned} \tag{2.11}$$

Clearly I^{++}, I^{--} coincide with the causal and future and past sets of S in $\emptyset$.

3. Construction of the Hawking vector-field in the causal region

We construct first the Killing vector-field $\mathbf{K}$ in the causal region $I^{++} \cup I^{--}$.

Proposition 3.1. *Under the assumptions of Theorem 1.1, there is a small neighborhood $\emptyset$ of S, a smooth Killing vector-field $\mathbf{K}$ in $\emptyset \cap \big(I^{++} \cup I^{--}\big)$ such that*

$$\mathbf{K} = \underline{u}L - u\,\underline{L} \quad \text{on } (\mathcal{N} \cup \underline{\mathcal{N}}) \cap \emptyset. \tag{3.1}$$

Moreover, in the region $\emptyset \cap (I^{++} \cup I^{--})$ where $\mathbf{K}$ is defined, $[\,\underline{L}, \mathbf{K}] = -\,\underline{L}$.

The rest of this section is concerned with the proof of Proposition 3.1. The first part of the proposition, which depends on the main assumption that the surfaces $\mathcal{N}$ and $\underline{\mathcal{N}}$ are non-expanding, is well known, see [7, Proposition B.1.]. For the sake of completeness, we provide its proof below.

Following [7] we construct the smooth vector-field $\mathbf{K}$ as the solution to the characteristic initial-value problem,

$$\square_{\mathbf{g}}\mathbf{K}=0, \qquad \mathbf{K}=\underline{u}L-u\,\underline{L} \quad \text{on } (\mathcal{N}\cup\underline{\mathcal{N}})\cap\emptyset. \tag{3.2}$$

As well known, see [18], the characteristic initial value problem for wave equations of type (3.3) is well posed. Thus the vector-field $\mathbf{K}$ is well-defined and smooth in the domain of dependence of $\mathcal{N}\cup\underline{\mathcal{N}}$ in $\emptyset$. Let $\pi_{\alpha\beta}={}^{(\mathbf{K})}\pi_{\alpha\beta}=\mathbf{D}_\alpha\mathbf{K}_\beta+\mathbf{D}_\beta\mathbf{K}_\alpha$. We have to prove that $\pi=0$ in a neighborhood of S intersected to $I^{++}\cup I^{--}$. It follows from (3.2), using the Bianchi identities and the Einstein vacuum equations, that π verifies the covariant wave equation,

$$\square_{\mathbf{g}}\pi_{\alpha\beta}=2\mathbf{R}^{\mu}{}_{\alpha\beta}{}^{\nu}\pi_{\mu\nu}. \tag{3.3}$$

In view of the standard uniqueness result for characteristic initial value problems, see [18], the statement of the proposition reduces to showing that $\pi=0$ on $(\mathcal{N}\cup\underline{\mathcal{N}})\cap\emptyset$. By symmetry, it suffices to prove that $\pi=0$ on $\mathcal{N}\cap\emptyset$. The proof relies on our main hypothesis, that the surfaces $\mathcal{N}$ and $\underline{\mathcal{N}}$ are non-expanding.

Since $\mathbf{K}=\underline{u}L$ on $\mathcal{N}\cap\emptyset$ is tangent to the null generators of $\mathcal{N}$, it follows that

$$\mathbf{D}_4\mathbf{K}_3=-1, \quad \mathbf{D}_4\mathbf{K}_4=\mathbf{D}_a\mathbf{K}_4=\mathbf{D}_4\mathbf{K}_a=\mathbf{D}_a\mathbf{K}_b=0, \quad a,b=1,2. \tag{3.4}$$

Thus, on $\mathcal{N}\cap\emptyset$

$$\pi_{44}=\pi_{a4}=\pi_{ab}=0 \quad a,b=1,2. \tag{3.5}$$

To prove that the remaining components of π vanish we use the wave equation $\square_{\mathbf{g}}\mathbf{K}=0$, which gives

$$\mathbf{D}_3\mathbf{D}_4\mathbf{K}_\mu+\mathbf{D}_4\mathbf{D}_3\mathbf{K}_\mu=\sum_{a=1}^{2}\mathbf{D}_a\mathbf{D}_a\mathbf{K}_\mu \quad \text{on } \mathcal{N}\cap\emptyset.$$

Since $\mathbf{D}_3\mathbf{D}_4\mathbf{K}_\mu-\mathbf{D}_4\mathbf{D}_3\mathbf{K}_\mu=\mathbf{R}_{34\mu\nu}\mathbf{K}^\nu$ on $\mathcal{N}\cap\emptyset$ (using (2.7)), we derive

$$2\mathbf{D}_4\mathbf{D}_3\mathbf{K}_\mu=\sum_{a=1}^{2}\mathbf{D}_a\mathbf{D}_a\mathbf{K}_\mu-\mathbf{R}_{34\mu\nu}\mathbf{K}^\nu,\ \mu=1,2,3,4,\ \text{on } \mathcal{N}\cap\emptyset. \tag{3.6}$$

We set first $\mu=4$. It follows from (3.4) that $\mathbf{D}_4\mathbf{D}_3\mathbf{K}_4=0$. In addition, $\mathbf{D}_3\mathbf{K}_4=1$ on S (the analogue of the first identity in (3.4) along the hypersurface $\underline{\mathcal{N}}$). Using (2.8) and (3.4), $\mathbf{D}_4\mathbf{D}_3\mathbf{K}_4=L(\mathbf{D}_3\mathbf{K}_4)$. Thus $\mathbf{D}_3\mathbf{K}_4=1$ on $\mathcal{N}$, which implies

$$\pi_{34}=0 \quad \text{on } \mathcal{N}. \tag{3.7}$$

We use now the equation (3.6) with $\mu=a\in\{1,2\}$ to calculate $P_a:=\pi_{a3}$ along $\mathcal{N}$. It follows from (3.4) and (2.7) that $\mathbf{D}_a\mathbf{D}_b\mathbf{K}_c=0$, $a,b,c=1,2$, and $\mathbf{R}_{34a\nu}\mathbf{K}^\nu=0$ on $\mathcal{N}$. A simple computation shows that $\mathbf{D}_a\mathbf{K}_3=\underline{u}\zeta_a$, thus $P_a=\mathbf{D}_3\mathbf{K}_a+\underline{u}\zeta_a$. Thus, using (2.8), $\mathbf{D}_3\mathbf{K}_4=1$, and $\mathbf{D}_b\mathbf{K}_c=0$ on $\mathcal{N}$, we derive

$$\begin{aligned}0=\mathbf{K}_{b;34}&=e_4(\mathbf{K}_{b;3})-\mathbf{K}_{\mathbf{D}_{e_4}e_b;e_3}-\mathbf{K}_{e_b;\mathbf{D}_{e_4}e_3}=e_4(P_b-\underline{u}\zeta_b)-\mathbf{K}_{\nabla_4e_b;e_3}+\zeta_bK_{e_4;e_3}\\&=\nabla_4(P_b-\underline{u}\zeta_b)+\zeta_b=\nabla_4P_b-\underline{u}\nabla_4\zeta_b.\end{aligned}$$

Thus

$$\nabla_4 P_a = \underline{u} \nabla_4 \zeta_a \quad \text{ on } \mathcal{N}.$$

On the other hand, along $\mathcal{N}$, ζ verifies the transport equation,

$$\nabla_4 \zeta_a = -\mathbf{R}_{a434} = 0.$$

Therefore, along $\mathcal{N}$,

$$\nabla_4 P_a = 0.$$

Since $P_a = \pi_{a3} = 0$ on S it follows that

$$\pi_{a3} = 0 \quad \text{ on } \mathcal{N}. \tag{3.8}$$

Similarly, denoting $Q = \pi_{33} = 2\mathbf{D}_3\mathbf{K}_3$, we have, according to (3.6) with $\mu = 3$,

$$\mathbf{D}_4\mathbf{D}_3\mathbf{K}_3 = \frac{1}{2}\big(\sum_{a=1}^{2} \mathbf{D}_a\mathbf{D}_a\mathbf{K}_3 - \rho\underline{u}\big), \qquad \rho = \mathbf{R}_{3434}. \tag{3.9}$$

Now, since we already now that π_{3b} vanishes on $\mathcal{N}$,

$$\mathbf{K}_{3;34} = e_4(\mathbf{K}_{3;3}) - \mathbf{K}_{\mathbf{D}_{e_4}e_3;e_3} - \mathbf{K}_{e_3;\mathbf{D}_{e_4}e_3} = \frac{1}{2}e_4(Q) + \sum_{b=1}^{2} \zeta_b \pi_{3b} = \frac{1}{2}e_4(Q). \tag{3.10}$$

On the other hand, using (2.8), $\mathbf{K}_{3;4} = -1$, $\mathbf{K}_{a;b} = 0$, and $\mathbf{K}_{3;a} = \underline{u}\zeta_a$,

$$\begin{aligned}\mathbf{K}_{3;ab} &= e_b(\mathbf{K}_{3;a}) - \mathbf{K}_{e_3;\mathbf{D}_{e_b}e_a} - \mathbf{K}_{\mathbf{D}_{e_b}e_3;e_a}\\ &= \partial_b(\underline{u}\zeta_a) - \underline{\chi}_{ba}\mathbf{K}_{3;4} - \zeta_b\mathbf{K}_{3;a}\\ &= \partial_b(\underline{u}\zeta_a) + \underline{\chi}_{ba} - \underline{u}\zeta_a\zeta_b,\end{aligned}$$

thus

$$\sum_{a=1}^{2} \mathbf{D}_a\mathbf{D}_a\mathbf{K}_3 \quad = \underline{u}(\mathrm{d}iv\, \zeta - |\zeta|^2) + \mathrm{tr}\, \underline{\chi}. \tag{3.11}$$

Therefore, equation (3.9) takes the form

$$L(Q) \quad = \quad \mathrm{tr}\, \underline{\chi} + \underline{u}(\mathrm{d}iv\, \zeta - |\zeta|^2 - \rho). \tag{3.12}$$

On the other hand we have the following structure equation on $\mathcal{N}$,

$$L(\mathrm{tr}\, \underline{\chi}) + \mathrm{d}iv\, \zeta - |\zeta|^2 - \rho = 0. \tag{3.13}$$

Thus, differentiating (3.12) with respect to L and applying (3.13) we derive,

$$\begin{aligned}L(L(Q)) \quad &= \quad L(\mathrm{tr}\, \underline{\chi}) + (\mathrm{d}iv\, \zeta - |\zeta|^2 - \rho) + \underline{u}L(\mathrm{d}iv\, \zeta - |\zeta|^2 - \rho)\\ &= \quad -\mathrm{d}iv\, \zeta + |\zeta|^2 + \rho + (\mathrm{d}iv\, \zeta - |\zeta|^2 - \rho) + \underline{u}L(\mathrm{d}iv\, \zeta - |\zeta|^2 - \rho).\end{aligned}$$

Using null structure equations, it is not hard to check that

$$L(\mathrm{d}iv\, \zeta) = L(|\zeta|^2) = L(\rho) = 0 \quad \text{ along } \mathcal{N}. \tag{3.14}$$

Indeed, the last identity follows from (2.7) and [14, Proposition 3.2.4]. The identity $L(|\zeta|^2) = 0$ follows from the transport equation $\nabla_4 \zeta_a = 0$. Therefore,

$$L(L(Q)) \quad = \quad 0 \qquad \text{along} \quad \mathcal{N}.$$

Since $L(Q) = 0$ on S (using again(3.12) restricted to S where both $\mathrm{tr}\,\underline{\chi}$ and $\underline{u}$ vanish), we infer that $L(Q) = 0$ along $\mathcal{N}$. Since $Q = 0$ on S we conclude that $Q = 0$ along $\mathcal{N}$ as desired. Thus $\pi_{33} = 0$, as desired.

The second part of the proposition, $[\underline{L}, \mathbf{K}] = -\underline{L}$ in a neighborhood of S in $I^{++} \cup I^{--}$, follows from the identity,

$$\mathbf{D}_{\underline{L}} W = -\mathbf{D}_W \underline{L} \quad \text{where} \quad W = [\underline{L}, \mathbf{K}] + \underline{L} = -\mathcal{L}_{\mathbf{K}} \underline{L} + \underline{L},$$

and the vanishing of W on $\mathcal{N} \cap \emptyset$. To prove the identity we make use of the fact that $\mathcal{L}_{\mathbf{K}}$ commutes with covariant differentiation. In particular, if $\mathbf{K}$ is Killing and X, Y arbitrary vector-fields then,

$$\mathcal{L}_{\mathbf{K}}(\mathbf{D}_X Y) \quad = \quad \mathbf{D}_X(\mathcal{L}_{\mathbf{K}} Y) + \mathbf{D}_{\mathcal{L}_{\mathbf{K}} X} Y. \tag{3.15}$$

Therefore,

$$\mathbf{D}_{\underline{L}} W \quad = \quad \mathbf{D}_{\underline{L}}(-\mathcal{L}_{\mathbf{K}} \underline{L} + \underline{L}) = -\mathbf{D}_{\underline{L}} \mathcal{L}_{\mathbf{K}} \underline{L} = \mathcal{L}_{\mathbf{K}}(\mathbf{D}_{\underline{L}} \underline{L}) + \mathbf{D}_{(\mathcal{L}_{\mathbf{K}} \underline{L})} \underline{L} = -\mathbf{D}_W \underline{L}.$$

as stated. It remains to prove that

$$W = [\underline{L}, \mathbf{K}] + \underline{L} = 0 \qquad \text{on } \mathcal{N} \cap \emptyset. \tag{3.16}$$

Since $\mathbf{K} = \underline{u} L$ on $\mathcal{N} \cap \emptyset$, this is equivalent to

$$\mathbf{D}_3 \mathbf{K}_\mu - \underline{u} \mathbf{D}_4 \underline{L}_\mu + \underline{L}_\mu = 0 \text{ on } \mathcal{N} \cap \emptyset, \quad \mu = 1, 2, 3, 4. \tag{3.17}$$

We check (3.17) on the null frame $e_1, e_2, e_3 = \underline{L}, e_4 = L$ defined earlier. The identity (3.17) follows for $\mu = a = 1, 2$ since $\mathbf{D}_3 \mathbf{K}_a = -\mathbf{D}_a \mathbf{K}_3 = -\underline{u} \zeta_a$, $\mathbf{D}_4 \underline{L}_a = \mathbf{g}(e_a, \mathbf{D}_{e_4} e_3) = -\zeta_a$ (see (2.8)), and $\underline{L}_a = 0$. The identity also follows for $\mu = 3$ since $\mathbf{D}_3 \mathbf{K}_3 = \pi_{33}/2 = 0$ (in view of Proposition 3.1), $\mathbf{D}_4 \underline{L}_3 = \mathbf{g}(e_3, \mathbf{D}_{e_4} e_3) = 0$ (see (2.8)), and $\underline{L}_3 = 0$. Finally, for $\mu = 4$, $\mathbf{D}_3 \mathbf{K}_4 = -\mathbf{D}_4 \mathbf{K}_3 = 1$ (see (3.4)), $\mathbf{D}_4 \underline{L}_4 = \mathbf{g}(e_4, \mathbf{D}_{e_4} e_3) = 0$, and $\underline{L}_4 = -1$. This completes the proof of the proposition.

4. Extension of the Hawking vector-field to a full neighborhood

In the previous section we have defined our Hawking vector-field $\mathbf{K}$ in a neighborhood $\emptyset$ of S intersected with $I^{++} \cup I^{--}$. To extend $\mathbf{K}$ in the exterior region $I^{+-} \cup I^{-+}$ we cannot rely on solving equation (3.2); the characteristic initial value problem is ill posed in that region. We need to rely instead on a completely different strategy, sketched in the introduction. We extend $\mathbf{K}$ by Lie dragging it relative to $\underline{L}$ and show that, for small $|t|$, $\Psi_t^* \mathbf{g}$ must coincide with $\mathbf{g}$, where $\Psi_t = \Psi_{t,\mathbf{K}}$ is the flow generated by $\mathbf{K}$. We show that both metrics coincide on $\mathcal{N} \cup \underline{\mathcal{N}}$ and, since they both verify the vacuum Einstein equations, we prove that the must coincide in a full neighborhood of S.

To implement this strategy we first define the vector-field K' by setting $K' = \underline{u} L$ on $\mathcal{N} \cap \emptyset$ and solving the ordinary differential equation $[\underline{L}, K'] = -\underline{L}$. The vector-field K'

is well-defined and smooth in a small neighborhood of S (since $\underline{L} \neq 0$ on S) and coincides with $\mathbf{K}$ in $I^{++} \cup I^{--}$ in $\emptyset$. Thus $\mathbf{K} := K'$ defines the desired extension. This proves the following.

Lemma 4.1. *There exists a smooth extension of the vector-field* $\mathbf{K}$ *(defined in Proposition 3.1) to an open neighborhood* $\emptyset$ *of* S *such that*

$$[\,\underline{L}, \mathbf{K}] = -\,\underline{L} \quad \text{in } \emptyset. \tag{4.1}$$

It remains to prove that $\mathbf{K}$ is indeed our desired Killing vector-field. For $|t|$ sufficiently small, we define, in a small neighborhood of S, the map $\Psi_t = \Psi_{t,\mathbf{K}}$ obtained by flowing a parameter distance t along the integral curves of $\mathbf{K}$. Let

$$\mathbf{g}^t = \Psi_t^*(\mathbf{g}).$$

The Lorentz metrics $\mathbf{g}^t$ are well-defined in a small neighborhood of S, for $|t|$ sufficiently small. To show that $\mathbf{K}$ is Killing we need to show that in fact $\mathbf{g}^t = \mathbf{g}$. Since $\mathbf{K}$ is tangent to $(\mathcal{N} \cup \underline{\mathcal{N}}) \cap \emptyset$ and is Killing in $I^{++} \cup I^{--}$, we infer that $\mathbf{g}^t = \mathbf{g}$ in a small neighborhood of S intersected with $I^{++} \cup I^{--}$. In view of the definition of $\mathbf{K}$ (see (4.1)),

$$\frac{d}{dt}\Psi_t^*\,\underline{L} \;=\; \lim_{h\to 0}\frac{\Psi_{t-h}^*\,\underline{L} - \Psi_t^*\,\underline{L}}{-h} = -\Psi_t^*\Big(\lim_{h\to 0}\frac{\Psi_{-h}^*\,\underline{L} - \Psi_0^*\,\underline{L}}{h}\Big) = -\Psi_t^*(\mathcal{L}_{\mathbf{K}}\,\underline{L}) = -\Psi_t^*\,\underline{L}.$$

We infer that,

$$\Psi_t^*\,\underline{L} = e^{-t}\,\underline{L}.$$

Now, given arbitrary vector-fields X, Y, we have $\mathbf{D}^t_{X^t} Y^t = \Psi_t^*(\mathbf{D}_X Y)$ where $\mathbf{D}^t$ denotes the covariant derivative induced by the metric $\mathbf{g}^t = \Psi_t^* g$ and $X^t = \Psi_t^* X$, $Y^t = \Psi_t^* Y$. In, particular $0 = \mathbf{D}^t_{\underline{L}^t}\,\underline{L}^t = e^{-2t}\mathbf{D}^t_{\underline{L}}\,\underline{L}$. This proves the following.

Lemma 4.2. *Assume* $\mathbf{K}$ *is a smooth vector-field verifying* (4.1) *and* $\mathbf{D}^t$ *the covariant derivative induced by the metric* $\mathbf{g}^t = \Psi_t^* g$. *Then,*

$$\mathbf{D}^t{}_{\underline{L}}\,\underline{L} = 0 \quad \text{in a small neighborhood of } S.$$

To summarize we have a family of metrics $\mathbf{g}^t$ which verify the Einstein vacuum equations $\mathbf{Ric}(\mathbf{g}^t) = 0$, $\mathbf{g}^t = \mathbf{g}$ in a small neighborhood of S intersected with $I^{++} \cup I^{--}$, and such that $\mathbf{D}^t{}_{\underline{L}}\,\underline{L} = 0$. Without loss of generality we may assume that both relations hold in $\emptyset$. Thus Theorem 1.1 is an immediate consequence of the following:

Proposition 4.3. *Assume* $\mathbf{g}'$ *is a smooth Lorentz metric on* $\emptyset$, *such that* $(\emptyset, \mathbf{g}')$ *is a smooth Einstein vacuum space-time. Assume that*

$$\mathbf{g}' = \mathbf{g} \quad \text{in} \;\; (I^{++} \cup I^{--}) \cap \emptyset \qquad \text{and} \qquad \mathbf{D}'_{\underline{L}}\,\underline{L} = 0 \;\text{ in } \emptyset,$$

where $\mathbf{D}'$ *denotes the covariant derivative induced by the metric* $\mathbf{g}'$. *Then* $\mathbf{g}' = \mathbf{g}$ *in a small neighborhood* $\emptyset' \subset \emptyset$ *of* S.

As explained in the introduction, this proposition was first proved in [1]. We provide here a more direct, simpler proof, specialized to our situation and based on the uniqueness result in Lemma 4.4 below. That lemma is an extension of the uniqueness results proved in [10] to coupled systems of covariant wave equations and ODE's. The motivation for the proof below was given in the introduction.

Proof of Proposition 4.3. It suffices to prove the proposition in a neighborhood $\emptyset(x_0)$ of a point x_0 in S in which we can introduce a fixed coordinate system x^α. Without loss of generality we may assume that

$$\mathbf{g}_{ij}(x_0) = \mathrm{diag}(-1,1,1,1), \qquad \sup_{x\in\emptyset(x_0)} \sum_{j=0}^{6} |\partial^j \mathbf{g}(x)| \leq A, \tag{4.2}$$

with $|\partial^j \mathbf{g}|$ denoting the sum of the absolute values of all partial derivatives of order j for all components of $\mathbf{g}$ in the given coordinate system. We may also assume, for the optical functions $u, \underline{u}$ introduced in section 2,

$$\sup_{x\in\emptyset(x_0)} (|\partial^j u(x)| + |\partial^j \underline{u}(x)|) \leq C_1 = C_1(A) \quad \text{ for } j = 0,\ldots,4. \tag{4.3}$$

In the rest of the proof we will keep restricting to smaller and smaller neighborhoods of x_0; for simplicity of notation we keep denoting such neighborhoods by $\emptyset(x_0)$.

Consider now our old null frame $\widetilde{v}_{(1)} = e_1, \widetilde{v}_{(2)} = e_2, \widetilde{v}_{(3)} = L, \widetilde{v}_{(4)} = \underline{L}$ on $\mathcal{N}\cap\emptyset(x_0)$ and define the vector-fields $v_{(1)}$, $v_{(2)}$,$v_{(3)}$, $v_{(4)} = \underline{L}$ and $v'_{(1)}$, $v'_{(2)}$, $v'_{(3)}$, $v'_{(4)} = \underline{L}$ by parallel transport along $\underline{L}$:

$$\begin{aligned}&\mathbf{D}_{\underline{L}} v_{(a)} = 0 \text{ and } v_{(a)} = \widetilde{v}_a \text{ on } \mathcal{N}\cap\emptyset(x_0);\\ &\mathbf{D}'_{\underline{L}} v'_{(a)} = 0 \text{ and } v'_{(a)} = \widetilde{v}_a \text{ on } \mathcal{N}\cap\emptyset(x_0).\end{aligned}$$

The vector-fields $v_{(a)}$ and $v'_{(a)}$ are well-defined and smooth in $\emptyset(x_0)$. Let $\mathbf{g}_{(a)(b)} = \mathbf{g}(v_{(a)}, v_{(b)})$, $\mathbf{g}'_{(a)(b)} = \mathbf{g}'(v'_{(a)}, v'_{(b)})$. The identities $\mathbf{D}_{\underline{L}} v_{(a)} = \mathbf{D}'_{\underline{L}} v'_{(a)} = 0$ show that $\underline{L}(\mathbf{g}_{(a)(b)}) = \underline{L}(\mathbf{g}'_{(a)(b)}) = 0$. Since $\mathbf{g}_{(a)(b)} = \mathbf{g}'_{(a)(b)}$ along $\mathcal{N}$ it follows that

$$\mathbf{g}_{(a)(b)} = \mathbf{g}'_{(a)(b)} := h_{(a)(b)} \text{ and } \underline{L}(h_{(a)(b)}) = 0 \text{ in } \emptyset(x_0). \tag{4.4}$$

For $a, b, c = 1, \ldots 4$ let

$$\begin{aligned}&\Gamma_{(a)(b)(c)} = \mathbf{g}(v_{(a)}, \mathbf{D}_{v_{(c)}} v_{(b)}), \qquad \Gamma'_{(a)(b)(c)} = \mathbf{g}'(v'_{(a)}, \mathbf{D}'_{v'_{(c)}} v'_{(b)}),\\ &(d\Gamma)_{(a)(b)(c)} = \Gamma'_{(a)(b)(c)} - \Gamma_{(a)(b)(c)}.\end{aligned}$$

For $a, b, c, d = 1, \ldots, 4$ let

$$\begin{aligned}&\mathbf{R}_{(a)(b)(c)(d)} = \mathbf{R}(v_{(a)}, v_{(b)}, v_{(c)}, v_{(d)}), \qquad \mathbf{R}'_{(a)(b)(c)(d)} = \mathbf{R}'(v'_{(a)}, v'_{(b)}, v'_{(c)}, v'_{(d)}),\\ &(dR)_{(a)(b)(c)(d)} = \mathbf{R}'_{(a)(b)(c)(d)} - \mathbf{R}_{(a)(b)(c)(d)}.\end{aligned}$$

Clearly, $\Gamma_{(a)(b)(4)} = \Gamma'_{(a)(b)(4)} = 0$. We use now the definition of the Riemann curvature tensor to find a system of equations for $\underline{L}[(d\Gamma)_{(a)(b)(c)}]$. We have

$$\begin{aligned}\mathbf{R}_{(a)(b)(c)(d)} &= \mathbf{g}(v_{(a)}, \mathbf{D}_{v_{(c)}}(\mathbf{D}_{v_{(d)}}v_{(b)}) - \mathbf{D}_{v_{(d)}}(\mathbf{D}_{v_{(c)}}v_{(b)}) - \mathbf{D}_{[v_{(c)},v_{(d)}]}v_{(b)})\\ &= \mathbf{g}(v_{(a)}, \mathbf{D}_{v_{(c)}}(\mathbf{g}^{(m)(n)}\Gamma_{(m)(b)(d)}v_{(n)})) - \mathbf{g}(v_{(a)}, \mathbf{D}_{v_{(d)}}(\mathbf{g}^{(m)(n)}\Gamma_{(m)(b)(c)}v_{(n)}))\\ &+ \mathbf{g}^{(m)(n)}\Gamma_{(a)(b)(n)}(\Gamma_{(m)(c)(d)} - \Gamma_{(m)(d)(c)})\\ &= v_{(c)}(\Gamma_{(a)(b)(d)}) - v_{(d)}(\Gamma_{(a)(b)(c)}) + \mathbf{g}^{(m)(n)}\Gamma_{(a)(b)(n)}(\Gamma_{(m)(c)(d)} - \Gamma_{(m)(d)(c)})\\ &+ \mathbf{g}_{(a)(n)}[\Gamma_{(m)(b)(d)}v_{(c)}(\mathbf{g}^{(m)(n)}) - \Gamma_{(m)(b)(c)}v_{(d)}(\mathbf{g}^{(m)(n)})]\\ &+ \mathbf{g}^{(m)(n)}(\Gamma_{(m)(b)(d)}\Gamma_{(a)(n)(c)} - \Gamma_{(m)(b)(c)}\Gamma_{(a)(n)(d)}).\end{aligned}$$

We set $d = 4$ and use $\Gamma_{(a)(b)(4)} = v_{(4)}(\mathbf{g}^{(a)(b)}) = 0$ and $\mathbf{g}^{(a)(b)} = h^{(a)(b)}$; the result is

$$\underline{L}(\Gamma_{(a)(b)(c)}) = -h^{(m)(n)}\Gamma_{(a)(b)(n)}\Gamma_{(m)(4)(c)} - \mathbf{R}_{(a)(b)(c)(4)}.$$

Similarly,

$$\underline{L}(\Gamma'_{(a)(b)(c)}) = -h^{(m)(n)}\Gamma'_{(a)(b)(n)}\Gamma'_{(m)(4)(c)} - \mathbf{R}'_{(a)(b)(c)(4)}.$$

We subtract these two identities to derive

$$\underline{L}[(d\Gamma)_{(a)(b)(c)})] = {}^{(1)}F^{(d)(e)(f)}_{(a)(b)(c)}(d\Gamma)_{(d)(e)(f)} - (dR)_{(a)(b)(c)(4)} \tag{4.5}$$

for some smooth function ${}^{(1)}F$. This can be written schematically in the form

$$\underline{L}(d\Gamma) = \mathcal{M}_\infty(d\Gamma) + \mathcal{M}_\infty(dR). \tag{4.6}$$

We will use such schematic equations for simplicity of notation[4].

For $a, b, c = 1, \ldots, 4$ and $\alpha = 0, \ldots, 3$ we define

$$\begin{aligned}(\partial d\Gamma)_{\alpha(a)(b)(c)} &= \partial_\alpha[(d\Gamma)_{(a)(b)(c)}];\\ (\partial dR)_{\alpha(a)(b)(c)(d)} &= \partial_\alpha[(dR)_{(a)(b)(c)(d)}],\end{aligned}$$

where ∂_α are the coordinate vector-fields relative to our local coordinates in $\emptyset(x_0)$. By differentiating (4.6),

$$\underline{L}(\partial d\Gamma) = \mathcal{M}_\infty(d\Gamma) + \mathcal{M}_\infty(\partial d\Gamma) + \mathcal{M}_\infty(dR) + \mathcal{M}_\infty(\partial dR). \tag{4.7}$$

Assume now that

$$\begin{aligned}v_{(a)} &= v^\alpha_{(a)}\partial_\alpha, \qquad v'_{(a)} = v'^\alpha_{(a)}\partial_\alpha,\\ v'_{(a)} - v_{(a)} &= (dv)^\alpha_{(a)}\partial_\alpha, \quad (dv)^\alpha_{(a)} = v'^\alpha_{(a)} - v^\alpha_{(a)},\end{aligned}$$

are the representations of the vectors $v_{(a)}$, $v'_{(a)}$, and $v'_{(a)} - v_{(a)}$ in our coordinate frame $\{\partial_\alpha\}_{\alpha=0,\ldots,3}$. Since $[v_{(4)}, v_{(b)}] = -\mathbf{D}_{v_{(b)}}v_{(4)} = -\Gamma^{(c)}{}_{(4)(b)}v_{(c)}$, we have

$$v^\alpha_{(4)}\partial_\alpha(v^\beta_{(b)}) - v^\alpha_{(b)}\partial_\alpha(v^\beta_{(4)}) = -\Gamma_{(a)(4)(b)}v^\beta_{(c)}\mathbf{g}^{(a)(c)}.$$

[4]In general, given $B = (B_1, \ldots B_L) : \emptyset(x_0) \to \mathbb{R}^L$ we let $\mathcal{M}_\infty(B) : \emptyset(x_0) \to \mathbb{R}^{L'}$ denote vector-valued functions of the form $\mathcal{M}_\infty(B)_{l'} = \sum_{l=1}^L A^l_{l'}B_l$, where the coefficients $A^l_{l'}$ are smooth on $\emptyset(x_0)$.

Similarly,

$$v_{(4)}^{\alpha}\partial_{\alpha}(v'^{\beta}_{(b)})-v'^{\alpha}_{(b)}\partial_{\alpha}(v^{\beta}_{(4)})=-\Gamma'_{(a)(4)(b)}v'^{\beta}_{(c)}\mathbf{g}'^{(a)(c)}.$$

We subtract these two identities to conclude that, schematically,

$$\underline{L}(dv)=\mathcal{M}_{\infty}(d\Gamma)+\mathcal{M}_{\infty}(dv). \tag{4.8}$$

As before, we define

$$(\partial dv)^{\beta}_{\alpha(b)}=\partial_{\alpha}[(dv)^{\beta}_{(b)}].$$

By differentiating (4.8) we have

$$\underline{L}(\partial dv)=\mathcal{M}_{\infty}(d\Gamma)+\mathcal{M}_{\infty}(\partial d\Gamma)+\mathcal{M}_{\infty}(dv)+\mathcal{M}_{\infty}(\partial dv). \tag{4.9}$$

Finally, we derive a wave equation for dR. We start from the identity

$$(\square_{\mathbf{g}}\mathbf{R})_{(a)(b)(c)(d)}-(\square_{\mathbf{g}'}\mathbf{R}')_{(a)(b)(c)(d)}=\mathcal{M}_{\infty}(dR),$$

which follows from the standard wave equations satisfied by $\mathbf{R}$ and $\mathbf{R}'$ and the fact that $\mathbf{g}^{(m)(n)}=\mathbf{g}'^{(m)(n)}=h^{(m)(n)}$. We also have

$$\begin{aligned}&\mathbf{D}_{(m)}\mathbf{R}_{(a)(b)(c)(d)}-\mathbf{D}'_{(m)}\mathbf{R}'_{(a)(b)(c)(d)}\\&=\mathcal{M}_{\infty}(dv)+\mathcal{M}_{\infty}(d\Gamma)+\mathcal{M}_{\infty}(d\mathbf{R})+\mathcal{M}_{\infty}(\partial d\mathbf{R}).\end{aligned}$$

It follows from the last two equations that

$$\begin{aligned}&\mathbf{g}^{(m)(n)}v_{(n)}(v_{(m)}(\mathbf{R}_{(a)(b)(c)(d)}))-\mathbf{g}'^{(m)(n)}v'_{(n)}(v'_{(m)}(\mathbf{R}'_{(a)(b)(c)(d)}))\\&=\mathcal{M}_{\infty}(dv)+\mathcal{M}_{\infty}(d\Gamma)+\mathcal{M}_{\infty}(\partial d\Gamma)+\mathcal{M}_{\infty}(dR)+\mathcal{M}_{\infty}(\partial dR).\end{aligned}$$

Since $\mathbf{g}^{(m)(n)}=\mathbf{g}'^{(m)(n)}$ it follows that

$$\begin{aligned}&\mathbf{g}^{(m)(n)}v_{(n)}(v_{(m)}((dR)_{(a)(b)(c)(d)}))\\&=\mathcal{M}_{\infty}(dv)+\mathcal{M}_{\infty}(\partial dv)+\mathcal{M}_{\infty}(d\Gamma)+\mathcal{M}_{\infty}(\partial d\Gamma)+\mathcal{M}_{\infty}(dR)+\mathcal{M}_{\infty}(\partial dR).\end{aligned}$$

Thus

$$\square_{\mathbf{g}}(dR)=\mathcal{M}_{\infty}(dv)+\mathcal{M}_{\infty}(\partial dv)+\mathcal{M}_{\infty}(d\Gamma)+\mathcal{M}_{\infty}(\partial d\Gamma)+\mathcal{M}_{\infty}(dR)+\mathcal{M}_{\infty}(\partial dR). \tag{4.10}$$

This is our main wave equation.

We collect now equations (4.6), (4.7), (4.8), (4.9), and (4.10):

$$\begin{cases}\underline{L}(d\Gamma)=\mathcal{M}_{\infty}(d\Gamma)+\mathcal{M}_{\infty}(dR);\\ \underline{L}(\partial d\Gamma)=\mathcal{M}_{\infty}(d\Gamma)+\mathcal{M}_{\infty}(\partial d\Gamma)+\mathcal{M}_{\infty}(dR)+\mathcal{M}_{\infty}(\partial dR);\\ \underline{L}(dv)=\mathcal{M}_{\infty}(dv)+\mathcal{M}_{\infty}(d\Gamma);\\ \underline{L}(\partial dv)=\mathcal{M}_{\infty}(dv)+\mathcal{M}_{\infty}(\partial dv)+\mathcal{M}_{\infty}(d\Gamma)+\mathcal{M}_{\infty}(\partial d\Gamma);\\ \square_{\mathbf{g}}(dR)=\mathcal{M}_{\infty}(dv)+\mathcal{M}_{\infty}(\partial dv)+\mathcal{M}_{\infty}(d\Gamma)+\mathcal{M}_{\infty}(\partial d\Gamma)+\mathcal{M}_{\infty}(dR)+\mathcal{M}_{\infty}(\partial dR).\end{cases} \tag{4.11}$$

This is our main system of equations. Since $\mathbf{g}=\mathbf{g}'$ in $I^{++}\cup I^{--}$, it follows easily that the functions $d\Gamma$, $\partial d\Gamma$, dv, ∂dv and dR vanish also in $I^{++}\cup I^{--}$. Therefore, the proposition follows from Lemma 4.4 below. □

Lemma 4.4. *Assume $G_i, H_j : \emptyset(x_0) \to \mathbb{R}$ are smooth functions, $i = 1, \dots, I$, $j = 1, \dots, J$. Let $G = (G_1, \dots, G_I)$, $H = (H_1, \dots, H_J)$, $\partial G = (\partial_0 G_1, \dots, \partial_4 G_I)$ and assume that in $\emptyset(x_0)$,*

$$\begin{cases} \Box_{\mathbf{g}} G = \mathcal{M}_\infty(G) + \mathcal{M}_\infty(\partial G) + \mathcal{M}_\infty(H); \\ \underline{L}(H) = \mathcal{M}_\infty(G) + \mathcal{M}_\infty(\partial G) + \mathcal{M}_\infty(H). \end{cases} \tag{4.12}$$

Assume that $G = 0$ and $H = 0$ on $(\mathcal{N} \cup \underline{\mathcal{N}}) \cap \emptyset(x_0)$. Then, there exists a small neighborhood $\emptyset'(x_0) \subset \emptyset(x_0)$ of x_0 such that $G = 0$ and $H = 0$ in $\big(I^{+-} \cup I^{-+}\big) \cap \emptyset'(x_0)$.

Unique continuation theorems of this type in the case $H = 0$ were proved by two of the authors in [10] and [11], using Carleman estimates. It is not hard to adapt the proofs, using similar Carleman estimates, to the general case; we provide all the details in the appendix. This completes the proof of Theorem 1.1.

We show now that the Killing vector-field $\mathbf{K}$ is timelike, in a quantitative sense, in a small neighborhood of S in the complement of $I^{++} \cup I^{--}$.

Proposition 4.5. *Let $\mathbf{K}$ be the Killing vector-field, constructed above, in a neighborhood $\emptyset$ of S. Then there is a neighborhood $\emptyset' \subset \emptyset$ of S such that*

$$\mathbf{g}(\mathbf{K}, \mathbf{K}) \leq u\underline{u} \quad \text{in } (I^{+-} \cup I^{-+}) \cap \emptyset'. \tag{4.13}$$

In particular, the vector-field $\mathbf{K}$ is timelike in the set $\emptyset' \setminus (I^{++} \cup I^{--})$.

Proof of Proposition 4.5. Since $\mathbf{K}$ is a Killing vector-field in $\emptyset$, we have

$$\Box_{\mathbf{g}}(\mathbf{K}^\beta \mathbf{K}_\beta) = 2\mathbf{D}^\alpha(\mathbf{K}^\beta \mathbf{D}_\alpha \mathbf{K}_\beta) = 2\mathbf{D}^\alpha \mathbf{K}^\beta \mathbf{D}_\alpha \mathbf{K}_\beta = -4 \quad \text{on } S. \tag{4.14}$$

Indeed, $\Box_{\mathbf{g}} \mathbf{K} = 0$ and it follows from (3.4) that $2\mathbf{D}^\alpha \mathbf{K}^\beta \mathbf{D}_\alpha \mathbf{K}_\beta = 4\mathbf{D}^3 \mathbf{K}^4 \mathbf{D}_3 \mathbf{K}_4 = -4$ on S. Since $\mathbf{K}_\beta \mathbf{K}^\beta = 0$ on $(\mathcal{N} \cup \underline{\mathcal{N}}) \cap \emptyset$ (see (3.1)), we have $\mathbf{K}_\beta \mathbf{K}^\beta = u\underline{u}f$ on $\emptyset$ for some smooth function $f : \emptyset \to \mathbb{R}$. Using (4.14) on S and the fact that $u = \underline{u} = 0$ on S, we derive

$$-4 = \mathbf{D}^\alpha \mathbf{D}_\alpha(u\underline{u}f) = 2f\mathbf{D}^\alpha u \mathbf{D}_\alpha \underline{u} = -2f\,\underline{L}(u)L(\underline{u}) = -2f.$$

Thus $f = 2$ on S, and the bound (4.13) follows for a sufficiently small $\emptyset'$. $\Box$

5. Further results in the presence of a symmetry

The goal of this section is to prove Theorem 1.2. So far we have constructed a smooth Killing vector-field $\mathbf{K}$ defined in an open set $\emptyset$ such that $\mathbf{K} = \underline{u}L - u\underline{L}$ on $(\mathcal{N} \cup \underline{\mathcal{N}}) \cap \emptyset$.

Assume in this section that the space-time $(\emptyset, \mathbf{g})$ admits another smooth Killing vector-field $\mathbf{T}$, which is tangent to the null hypersurfaces $\mathcal{N}$ and $\underline{\mathcal{N}}$. We recall several definitions (see [10, Section 4] for a longer discussion and proofs of some identities). In $\emptyset$ we define the 2-form $F_{\alpha\beta} = \mathbf{D}_\alpha \mathbf{T}_\beta$ and the complex valued 2-form,

$$\mathcal{F}_{\alpha\beta} = F_{\alpha\beta} + i\,{}^*F_{\alpha\beta} = F_{\alpha\beta} + (i/2)\in_{\alpha\beta}{}^{\mu\nu} F_{\mu\nu}. \tag{5.1}$$

Let $\mathcal{F}^2 = \mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta}$. We define also the Ernst 1-form

$$\sigma_\mu = 2\mathbf{T}^\alpha \mathcal{F}_{\alpha\mu} = \mathbf{D}_\mu(-\mathbf{T}^\alpha \mathbf{T}_\alpha) - i \in_{\mu\beta\gamma\delta} \mathbf{T}^\beta \mathbf{D}^\gamma \mathbf{T}^\delta. \tag{5.2}$$

It is easy to check that, in $\emptyset$

$$\begin{cases} \mathbf{D}_\mu \sigma_\nu - \mathbf{D}_\nu \sigma_\mu = 0; \\ \mathbf{D}^\mu \sigma_\mu = -\mathcal{F}^2; \\ \sigma_\mu \sigma^\mu = \mathbf{g}(\mathbf{T}, \mathbf{T}) \mathcal{F}^2. \end{cases} \tag{5.3}$$

Proposition 5.1. *There is an open set $\emptyset' \subseteq \emptyset$, $S \subseteq \emptyset'$ such that*

$$[\mathbf{T}, \mathbf{K}] = 0 \ \textit{in}\ \emptyset'. \tag{5.4}$$

In addition, if $\sigma_\mu = 2\mathbf{T}^\alpha \mathcal{F}_{\alpha\mu}$ is the Ernst 1-form associated to $\mathbf{T}$ (see (5.2)), then

$$\mathbf{K}^\mu \sigma_\mu = 0 \ \textit{in}\ \emptyset'. \tag{5.5}$$

Proof of Proposition 5.1. We show first that

$$[\mathbf{T}, \mathbf{K}] = 0 \quad \text{ on } (\mathcal{N} \cup \underline{\mathcal{N}}) \cap \emptyset. \tag{5.6}$$

By symmetry, it suffices to check that $[\mathbf{T}, \mathbf{K}] = 0$ on $\mathcal{N} \cap \emptyset$. We first observe that $[\mathbf{T}, L]$ is proportional to L. Indeed, since the null second fundamental form of $\mathcal{N}$ is symmetric and $\mathbf{T}$ is both Killing and tangent to $\mathcal{N}$, we have for every $X \in T(\mathcal{N})$,

$$\begin{aligned} \mathbf{g}([\mathbf{T}, L], X) &= \mathbf{g}(\mathbf{D}_\mathbf{T} L, X) - \mathbf{g}(\mathbf{D}_L \mathbf{T}, X) = \mathbf{g}(\mathbf{D}_\mathbf{T} L, X) + \mathbf{g}(\mathbf{D}_X \mathbf{T}, L) \\ &= \mathbf{g}(\mathbf{D}_\mathbf{T} L, X) - \mathbf{g}(\mathbf{T}, \mathbf{D}_X L) = \chi(\mathbf{T}, X) - \chi(X, \mathbf{T}) = 0. \end{aligned}$$

Consequently $[\mathbf{T}, L]$ must be proportional to L, i.e. $[\mathbf{T}, L] = fL$. Since $\mathbf{D}_L L = 0$ and $\mathbf{T}$ commutes with covariant derivatives we derive,

$$\begin{aligned} 0 &= \mathcal{L}_\mathbf{T}(\mathbf{D}_L L) = \mathbf{D}_{\mathcal{L}_T L} L + \mathbf{D}_L(\mathcal{L}_\mathbf{T} L) \\ &= \mathbf{D}_{fL} L + \mathbf{D}_L(fL) = L(f) L. \end{aligned}$$

Therefore

$$[\mathbf{T}, L] = fL \qquad \text{and} \qquad L(f) = 0 \qquad \text{on } \mathcal{N} \cap \emptyset. \tag{5.7}$$

On the other hand, in view of the definition of $\underline{u}$ we have $\mathbf{T}(L(\underline{u})) - L(\mathbf{T}(\underline{u})) = fL(\underline{u})$. Hence,

$$L(f\underline{u} + \mathbf{T}(\underline{u})) = 0.$$

Since $\mathbf{T}$ is tangent to S and $\underline{u} = 0$ on S, we deduce that $f\underline{u} + \mathbf{T}(\underline{u})$ vanishes on S, thus

$$\mathbf{T}\underline{u} + f\underline{u} = 0, \qquad \text{on } \mathcal{N} \cap \emptyset.$$

Now, $[\mathbf{T}, \underline{u}L] = \mathbf{T}(\underline{u})L + \underline{u}[\mathbf{T}, L] = \big(\mathbf{T}(\underline{u}) + f\underline{u}\big)L = 0$. The identity (5.6) follows since $\mathbf{K} = \underline{u}L$ on $\mathcal{N} \cap \emptyset$.

Let $V = [\mathbf{T}, \mathbf{K}] = \mathcal{L}_\mathbf{T}\mathbf{K}$ on $\emptyset$. Since $\square_\mathbf{g}\mathbf{K} = 0$ and $\mathbf{T}$ is Killing, we derive, after commuting covariant and Lie derivatives,

$$0 = \mathcal{L}_\mathbf{T}(\square_\mathbf{g}\mathbf{K}) = \square_\mathbf{g}(\mathcal{L}_\mathbf{T}\mathbf{K}) = \square_\mathbf{g} V.$$

Since V vanishes on $(\mathcal{N} \cup \underline{\mathcal{N}}) \cap \emptyset$, it follows that V vanishes in $(I^{++} \cup I^{--}) \cap \emptyset'$, for some smaller neighborhood $\emptyset'$ of S. (due to the well-posedness of the characteristic initial-value problem); it also follows that V vanishes in $(I^{+-}\cup I^{-+})\cap\emptyset'$ using Lemma 4.4 with $H=0$. This completes the proof of (5.4).

We prove now the identity (5.5). Since $\mathbf{K}$ and $\mathbf{T}$ commute we observe that $\mathcal{L}_{\mathbf{K}}\mathcal{F}=0$ in $\emptyset$. In addition, since $\square_{\mathbf{g}}\mathbf{K}=0$, $\mathbf{DK}$ is antisymmetric, $\mathbf{D}\sigma$ is symmetric with trace $\mathbf{D}^\alpha\sigma_\alpha=-\mathcal{F}^2$ (see (5.3)) and $\mathbf{Ric}(\mathbf{g})=0$, we have in $\emptyset$

$$\square_{\mathbf{g}}(\mathbf{K}^\mu\sigma_\mu)=\mathbf{K}^\mu\square_{\mathbf{g}}\sigma_\mu=\mathbf{K}^\mu\mathbf{D}_\mu(\mathbf{D}^\alpha\sigma_\alpha)=-\mathcal{L}_{\mathbf{K}}\mathcal{F}^2=0. \tag{5.8}$$

We show below that the function $\mathbf{K}^\mu\sigma_\mu$ vanishes on $(\mathcal{N}\cup\underline{\mathcal{N}})\cap\emptyset$. Thus, as before, we conclude that $\mathbf{K}^\mu\sigma_\mu=0$ in a smaler neighborhood $\emptyset'$, as desired.

To show that $\mathbf{K}^\mu\sigma_\mu$ vanishes on $(\mathcal{N}\cup\underline{\mathcal{N}})\cap\emptyset$ we calculate with respect to our null frame $L=e_4$, $\underline{L}=e_3$, e_1, e_2 defined in a neighborhood of S. Since $\mathbf{T}$ is tangent to $\mathcal{N}$, for $a=1,2$ we have $F_{a4}=e_a(\mathbf{g}(\mathbf{T},e_4))-\mathbf{g}(\mathbf{T},\mathbf{D}_{e_a}e_4)=0$ along $\mathcal{N}$ (since $\mathbf{D}_{e_a}e_4=-\zeta_a e_4$, see (2.8)). Similarly, $F_{a3}=0$ along $\underline{\mathcal{N}}$. Thus

$$\mathcal{F}_{14}=\mathcal{F}_{24}=0 \text{ on } \mathcal{N}\cap\emptyset \quad\text{ and }\quad \mathcal{F}_{13}=\mathcal{F}_{23}=0 \text{ on } \underline{\mathcal{N}}\cap\emptyset. \tag{5.9}$$

Since $\mathbf{K}=\underline{u}e_4-ue_3$ on $(\mathcal{N}\cup\underline{\mathcal{N}})\cap\emptyset$, we infer that,

$$\mathbf{K}^\mu\sigma_\mu=2\mathbf{K}^\mu\mathbf{T}^\alpha\mathcal{F}_{\alpha\mu}=0 \quad \text{ on } (\mathcal{N}\cup\underline{\mathcal{N}})\cap\emptyset, \tag{5.10}$$

as desired. □

Proposition 5.2. *There is a constant $\lambda_0\in\mathbb{R}$ and an open neighborhood $\emptyset'\subseteq\emptyset$ of S such that the vector-field*

$$\mathbf{Z}=\mathbf{T}+\lambda_0\mathbf{K}$$

has periodic orbits in $\emptyset'$. In other words , there is $t_0>0$ such that $\Psi_{t_0,\mathbf{Z}}=\mathrm{Id}$ in $\emptyset'$.

This completes the proof of Theorem 1.2. Observe that the main constants λ_0 and t_0 can be determined on the bifurcation sphere S. We show below that Proposition 5.2 follows from the following lemma.

Lemma 5.3. *There is a constant $t_0>0$ such that $\Psi_{t_0,\mathbf{T}}=\mathrm{Id}$ in S. In addition, there is a constant $\lambda_0\in\mathbb{R}$ and a choice of the null pair $(L,\underline{L})$ along S (satisfying (2.1)) such that*

$$[\mathbf{T},L]=\lambda_0 L \quad\textit{and}\quad [\mathbf{T},\underline{L}]=-\lambda_0\underline{L} \quad \textit{on } S. \tag{5.11}$$

Proof of Proposition 5.2. It follows from (5.7) and (5.11) that

$$[\mathbf{T},L]=\lambda_0 L \quad\text{on } \mathcal{N}\cap\emptyset \quad\text{ and }\quad [\mathbf{T},\underline{L}]=-\lambda_0\underline{L} \quad \text{on } \underline{\mathcal{N}}\cap\emptyset. \tag{5.12}$$

Thus, using the identity $[\underline{L},\mathbf{K}]=-\underline{L}$ in Proposition 3.1,

$$[\mathbf{Z},\underline{L}]=[\mathbf{T}+\lambda_0\mathbf{K},\underline{L}]=0 \quad \text{on } \underline{\mathcal{N}}\cap\emptyset.$$

Since $\mathbf{Z}$ is a Killing vector-field, it follows as in the proof of Proposition 3.1 (see (3.15)) that

$$[\mathbf{Z},\underline{L}]=0 \quad \text{in } \emptyset.$$

An argument similar to the proof of (3.16) shows that $[L,\mathbf{K}]-L=0$ on $\underline{\mathcal{N}}\cap\emptyset$. Using the first identity in (5.12), it follows that $[\mathbf{Z},L]=0$ on $\underline{\mathcal{N}}\cap\emptyset$. Since $\mathbf{Z}$ is a Killing vector-field, it follows as in Proposition 3.1 that $[\mathbf{Z},L]=0$ in $\emptyset$.

The conclusion of the proposition follows from the first claim in Lemma 5.3 and the identities $[\mathbf{Z}, \underline{L}] = [\mathbf{Z}, L] = 0$ in $\emptyset$. $\square$

Proof of Lemma 5.3. The existence of the period t_0 is a standard fact concerning Killing vector-fields on the sphere[5]. In particular all nontrivial orbits of S are compact and diffeomorphic to $\mathbb{S}^1$. To prove (5.11), in view of (5.7) it suffices to prove that there is $\lambda_0 \in \mathbb{R}$ and a choice of the null pair $(L, \underline{L})$ on S such that

$$\mathbf{g}([\mathbf{T}, L], \underline{L}) = -\lambda_0, \qquad \mathbf{g}([\mathbf{T}, \underline{L}], L) = \lambda_0 \qquad \text{on } S.$$

Both identities are equivalent to

$$\mathbf{T}^\alpha \underline{L}^\beta \mathbf{D}_\alpha L_\beta - L^\alpha \underline{L}^\beta \mathbf{D}_\alpha \mathbf{T}_\beta = -\lambda_0,$$

which is equivalent to

$$\lambda_0 = F_{43} - \mathbf{g}(\zeta, \mathbf{T}).$$

We thus have to show that there exist a choice of the null pair $e_4 = L, e_3 = \underline{L}$ along S such that the scalar function below is constant along S,

$$H := F_{43} - \mathbf{g}(\zeta, \mathbf{T}). \tag{5.13}$$

Under a scaling transformation $e_4' = f e_4, e_3' = f^{-1} e_3$ the torsion ζ changes according to the formula,

$$\zeta' = \zeta - \nabla \log f.$$

Therefore, in the new frame,

$$H' = F_{4'3'} - \mathbf{g}(\zeta', \mathbf{T}) = F_{43} - \mathbf{g}(\zeta, \mathbf{T}) + \mathbf{T}(\log f) = H + \mathbf{T}(\log f)$$

Consequently, we are led to look for a function f such that $H + \mathbf{T}(\log f)$ is a constant. Taking $\hat{H}$ to be the average of H along the integral curves of $\mathbf{T}$ and solving the equation

$$\mathbf{T}(\log f) \;\; = \;\; -H + \hat{H}, \tag{5.14}$$

it only remains to prove that $\hat{H}$ is constant along S.

Since $\mathbf{T}$ is Killing we must have,

$$\mathbf{D}_\alpha \mathbf{D}_\beta T_\gamma = T^\lambda \mathbf{R}_{\lambda\alpha\beta\gamma} \tag{5.15}$$

Using (5.15) and the formulas (2.8) on S we derive,

$$\mathbf{T}^\lambda \mathbf{R}_{\lambda a 43} \;\; = \;\; \mathbf{D}_a \mathbf{D}_4 \mathbf{T}_3 = e_a(\mathbf{D}_4 \mathbf{T}_3) = e_a(F_{43}).$$

Thus, since $\mathbf{T}$ is tangent to S and $\mathbf{T}^b \mathbf{R}_{ba43} = \frac{1}{2} \in_{ab} \mathbf{T}^b \sigma$ (with $\sigma = {}^*\mathbf{R}_{3434}$)

$$e_a(F_{43}) \;\; = \;\; \mathbf{T}^b \mathbf{R}_{ba43} = \frac{1}{2} \in_{ab} \mathbf{T}^b \sigma. \tag{5.16}$$

In particular, the function H defined in (5.13) is constant on S if $\mathbf{T} \equiv 0$ on S. Thus we may assume in the rest of the proof that the set $\Lambda = \{p \in S : \mathbf{T}_p = 0\}$ is finite.

[5]If $\mathbf{T} \equiv 0$ on S then any value of $t_0 > 0$ is suitable. In this case, the conclusion of Proposition 5.2 is that $\mathbf{T} + \lambda_0 \mathbf{K} \equiv 0$ in $\emptyset'$ for some $\lambda_0 \in \mathbb{R}$.

On the other hand, writing $\nabla_a \zeta_b - \nabla_b \zeta_a = \in_{ab} curl\, \zeta$,

$$\begin{aligned} e_a \mathbf{g}(\zeta, \mathbf{T}) &= \nabla_a \zeta_b \mathbf{T}^b + \zeta_b \nabla_a \mathbf{T}_b = (\nabla_a \zeta_b - \nabla_b \zeta_a)\mathbf{T}^b + \zeta^b \nabla_a \mathbf{T}_b + \nabla_{\mathbf{T}} \zeta_a \\ &= \in_{ab} curl\, \zeta \mathbf{T}^b + \zeta^b \nabla_a \mathbf{T}_b + \nabla_{\mathbf{T}} \zeta_a \end{aligned}$$

The torsion ζ verifies the equation,

$$curl\, \zeta = \frac{1}{2}\sigma, \tag{5.17}$$

Therefore,

$$e_a \mathbf{g}(\zeta, \mathbf{T}) = \frac{1}{2} \in_{ab} \mathbf{T}^b \sigma + \zeta^b \nabla_a \mathbf{T}_b + \nabla_{\mathbf{T}} \zeta_a. \tag{5.18}$$

Since $H = F_{43} - \zeta \underset{\smile}{\mathbf{T}}$ we deduce,

$$e_a(H) = -\zeta^b \nabla_a \mathbf{T}_b - \nabla_{\mathbf{T}} \zeta_a. \tag{5.19}$$

Consider the orthonormal frame e_1, e_2 on $S \setminus \Lambda$,

$$e_1 = X^{-1}\mathbf{T}, \qquad X^2 = \mathbf{g}(\mathbf{T}, \mathbf{T}).$$

Since $e_1(X) = 0$ and $e_1 = X^{-1}\mathbf{T}$, we have

$$\nabla_{\mathbf{T}} e_2 = -F_{12} e_1.$$

We claim that, with respect to this local frame,

$$\nabla_2(H) = -\mathbf{T}(\zeta_2). \tag{5.20}$$

Indeed,

$$\begin{aligned} \nabla_2(H) &= -\zeta^1 \nabla_2 \mathbf{T}_1 - \zeta^2 \nabla_2 \mathbf{T}_2 - \mathbf{g}(\nabla_{\mathbf{T}} \zeta, e_2) \\ &= -\zeta^1 F_{21} - \mathbf{T}\mathbf{g}(\zeta, e_2) + \mathbf{g}(\zeta, \nabla_{\mathbf{T}} e_2) \\ &= -\mathbf{T}\mathbf{g}(\zeta, e_2) - \zeta^1 F_{21} - \zeta^1 F_{12} \\ &= -\mathbf{T}(\zeta_2) \end{aligned}$$

We now fix a a non-trivial orbit γ_0 of $\mathbf{T}$ in $S \setminus \Lambda$. Consider the geodesics initiating on γ_0 and perpendicular to it and ϕ the corresponding affine parameter. More precisely we choose a vector V on γ_0 such that $\mathbf{g}(V, V) = 1$ and extend it by parallel transport along the geodesics perpendicular to γ_0. Then choose ϕ such that $V(\phi) = 1$ and $\phi = 0$ on γ_0. This defines a system of coordinates t, ϕ in a neighborhood U of γ_0, such that $\partial_t = T$, $\nabla_{\partial_\phi} \partial_\phi = 0$ in U and $\mathbf{g}(\partial_t, \partial_\phi) = 0$, $\mathbf{g}(\partial_\phi, \partial_\phi) = 1$ on Γ_0. Since ∂_t is Killing we must have $X^2 = -\mathbf{g}(\partial_t, \partial_t)$ and $\mathbf{g}(\partial_\phi, \partial_\phi)$ independent of t. Moreover,

$$\partial_\phi \mathbf{g}(\partial_t, \partial_\phi) = \mathbf{g}(\nabla_{\partial_\phi} \partial_t, \partial_\phi) + \mathbf{g}(\partial_t, \nabla_{\partial_\phi} \partial_\phi) = \mathbf{g}(\nabla_{\partial_t} \partial_\phi, \partial_\phi) = \frac{1}{2}\partial_t \mathbf{g}(\partial_\phi, \partial_\phi) = 0.$$

Hence, since $\mathbf{g}(\partial_t, \partial_\phi) = 0$ on Γ_0 we infer that $\mathbf{g}(\partial_t, \partial_\phi) = 0$ in U. Similarly,

$$\partial_\phi \mathbf{g}(\partial_\phi, \partial_\phi) = 2\mathbf{g}(\nabla_{\partial_\phi} \partial_\phi, \partial_\phi) = 0$$

and therefore, $\mathbf{g}(\partial_\phi, \partial_\phi) = 1$ in U. Thus, in U, the metric $\mathbf{g}$ takes the form,

$$d\phi^2 + X^2(\phi)dt^2 \tag{5.21}$$

Therefore, with $\mathbf{T} = \partial_t$, $e_2 = \partial_\phi$, we deduce from (5.20), everywhere in U,

$$\partial_\phi H = -\partial_t \mathbf{g}(\zeta, \partial_\phi) \tag{5.22}$$

Thus, integrating in t and in view of the fact that the orbits of ∂_t are closed, we infer that $\hat{H}$ is constant along S, as desired. □

Appendix A. Proof of Lemma 4.4

We will use a Carleman estimate proved by two of the authors in [10, Section 3], which we recall below. Let $\emptyset(x_0)$ a coordinate neighborhood of a point $x_0 \in S$ and coordinates x^α as in (4.2). We denote by $B_r = B_r(x_0)$, the set of points $p \in \emptyset(x_0)$ whose coordinates $x = x^\alpha$ verify $|x - x_0| \le r$, relative to the standard euclidean norm in $\emptyset(x_0)$. Consider two vector-fields $V = V^\alpha \partial_\alpha, W = W^\alpha \partial_\alpha$ on $\emptyset(x_0)$ which verify, that,

$$\sup_{x \in \emptyset(x_0)} \sum_{j=0}^{4} (|\partial^j V(x)| + |\partial^j W(x)|) \le A, \tag{A.1}$$

where A is a large constant (as in (4.2)), and $|\partial^j V(x)|$ denotes the sum of the absolute values of all partial derivatives of order j of all components of V in our given coordinate system. When $j = 1$ we write simply $|\partial V(x)|$.

Definition A.1. *A family of weights $h_\epsilon : B_{\epsilon^{10}} \to \mathbb{R}_+$, $\epsilon \in (0, \epsilon_1)$, $\epsilon_1 \le A^{-1}$, will be called V-conditional pseudo-convex if for any $\epsilon \in (0, \epsilon_1)$*

$$h_\epsilon(x_0) = \epsilon, \quad \sup_{x \in B_{\epsilon^{10}}} \sum_{j=1}^{4} \epsilon^j |\partial^j h_\epsilon(x)| \le \epsilon/\epsilon_1, \quad |V(h_\epsilon)(x_0)| \le \epsilon^{10}, \tag{A.2}$$

$$\mathbf{D}^\alpha h_\epsilon(x_0) \mathbf{D}^\beta h_\epsilon(x_0) (\mathbf{D}_\alpha h_\epsilon \mathbf{D}_\beta h_\epsilon - \epsilon \mathbf{D}_\alpha \mathbf{D}_\beta h_\epsilon)(x_0) \ge \epsilon_1^2, \tag{A.3}$$

and there is $\mu \in [-\epsilon_1^{-1}, \epsilon_1^{-1}]$ such that for all vectors $X = X^\alpha \partial_\alpha$ at x_0

$$\begin{aligned} &\epsilon_1^2 [(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2] \\ &\le X^\alpha X^\beta (\mu \mathbf{g}_{\alpha\beta} - \mathbf{D}_\alpha \mathbf{D}_\beta h_\epsilon)(x_0) + \epsilon^{-2} (|X^\alpha V_\alpha(x_0)|^2 + |X^\alpha \mathbf{D}_\alpha h_\epsilon(x_0)|^2). \end{aligned} \tag{A.4}$$

A function $e_\epsilon : B_{\epsilon^{10}} \to \mathbb{R}$ will be called a negligible perturbation if

$$\sup_{x \in B_{\epsilon^{10}}} |\partial^j e_\epsilon(x)| \le \epsilon^{10} \qquad \text{for } j = 0, \ldots, 4. \tag{A.5}$$

Our main Carleman estimate, see [10, Section 3], is the following:

Lemma A.2. *Assume $\epsilon_1 \le A^{-1}$, $\{h_\epsilon\}_{\epsilon \in (0, \epsilon_1)}$ is a V-conditional pseudo-convex family, and e_ϵ is a negligible perturbation for any $\epsilon \in (0, \epsilon_1]$. Then there is $\epsilon \in (0, \epsilon_1)$ sufficiently* *small (depending only on ϵ_1) and $\widetilde{C}_\epsilon$ sufficiently large such that for any $\lambda \geq \widetilde{C}_\epsilon$ and any $\phi \in C_0^\infty(B_{\epsilon^{10}})$*

$$\lambda \| e^{-\lambda f_\epsilon} \phi \|_{L^2} + \| e^{-\lambda f_\epsilon} |\partial \phi| \, \|_{L^2} \leq \widetilde{C}_\epsilon \lambda^{-1/2} \| e^{-\lambda f_\epsilon} \square_{\mathbf{g}} \phi \|_{L^2} + \epsilon^{-6} \| e^{-\lambda f_\epsilon} V(\phi) \|_{L^2}, \tag{A.6}$$

where $f_\epsilon = \ln(h_\epsilon + e_\epsilon)$.

We will only use this Carleman estimate with $V = 0$. In this case the pseudo-convexity condition in Definition A.1 is a special case of Hörmander's pseudo-convexity condition [9, Chapter 28]. We also need a Carleman estimate to exploit the ODE's in (4.12).

Lemma A.3. *Assume $\epsilon \leq A^{-1}$ is sufficiently small, e_ϵ is a negligible perturbation, and $h_\epsilon : B_{\epsilon^{10}} \to \mathbf{R}_+$ satisfies*

$$h_\epsilon(x_0) = \epsilon, \quad \sup_{x \in B_{\epsilon^{10}}} \sum_{j=1}^{2} \epsilon^j |\partial^j h_\epsilon(x)| \leq 1, \quad |W(h_\epsilon)(x_0)| \geq 1. \tag{A.7}$$

Then there is $\widetilde{C}_\epsilon$ sufficiently large such that for any $\lambda \geq \widetilde{C}_\epsilon$ and any $\phi \in C_0^\infty(B_{\epsilon^{10}})$

$$\| e^{-\lambda f_\epsilon} \phi \|_{L^2} \leq 4 \lambda^{-1} \| e^{-\lambda f_\epsilon} W(\phi) \|_{L^2}, \tag{A.8}$$

where $f_\epsilon = \ln(h_\epsilon + e_\epsilon)$.

Proof of Lemma A.3. Clearly, we may assume that ϕ is real-valued and let $\psi = e^{-\lambda f_\epsilon} \phi \in C_0^\infty(B_{\epsilon^{10}})$. We have to prove that

$$\| \psi \|_{L^2} \leq 4 \| \lambda^{-1} W(\psi) + W(f_\epsilon) \psi \|_{L^2}. \tag{A.9}$$

By integration by parts,

$$\begin{aligned} &\int_{B_{\epsilon^{10}}} [\lambda^{-1} W(\psi) + W(f_\epsilon)\psi] \cdot W(f_\epsilon) \psi \, d\mu \\ &= \int_{B_{\epsilon^{10}}} [W(f_\epsilon)\psi]^2 \, d\mu - (2\lambda)^{-1} \int_{B_{\epsilon^{10}}} \psi^2 \cdot \mathbf{D}_\alpha(W(f_\epsilon) W^\alpha) d\mu. \end{aligned}$$

In view of (A.7) and the assumption (A.1)

$$|W(f_\epsilon)| \geq 1 \quad \text{and} \quad |\mathbf{D}_\alpha(W(f_\epsilon) W^\alpha)| \leq \widetilde{C}_\epsilon \quad \text{in} \quad B_{\epsilon^{10}},$$

provided that ϵ is sufficiently small. Thus, for λ sufficiently large,

$$\int_{B_{\epsilon^{10}}} [\lambda^{-1} W(\psi) + W(f_\epsilon)\psi] \cdot W(f_\epsilon)\psi \, d\mu \geq \frac{1}{2} \int_{B_{\epsilon^{10}}} [W(f_\epsilon)\psi]^2 \, d\mu,$$

and the bound (A.9) follows. □

Proof of Lemma 4.4. It suffices to prove that $G = 0$ and $H = 0$ in $I_{\widetilde{c}}^{+-}$, for some $\widetilde{c}$ sufficiently small. We fix $x_0 \in S$ and set

$$h_\epsilon = \epsilon^{-1}(u + \epsilon)(-\underline{u} + \epsilon) \quad \text{and} \quad e_\epsilon = \epsilon^{10} N^{x_0}, \tag{A.10}$$

where $u, \underline{u}$ are the optical functions defined in section 2 and $N^{x_0}(x) = |x - x_0|^2 = \sum_{\alpha=0,1,2,3} |x^\alpha - x_0^\alpha|^2$, the square of the standard euclidean norm.

It is clear that e_ϵ is a negligible perturbation, in the sense of (A.5), for ϵ sufficiently small. Also, it is clear that h_ϵ verifies the condition (A.7), for ϵ sufficiently small and $W = 2\underline{L}$.

We show now that there is $\epsilon_1 = \epsilon_1(A)$ sufficiently small such that the family of weights $\{h_\epsilon\}_{\epsilon\in(0,\epsilon_1)}$ is 0-conditional pseudo-convex, in the sense of Definition A.1. Condition (A.2) is clearly satisfied, in view of the definition and (4.3). To verify conditions (A.3) and (A.4), we compute, in the frame e_1, e_2, e_3, e_4 defined in section 2,

$$e_1(h_\epsilon) = e_2(h_\epsilon) = 0, \quad e_3(h_\epsilon) = -\Omega(1 - \epsilon^{-1}\underline{u}), \quad e_4(h_\epsilon) = \Omega(1 + \epsilon^{-1}u) \tag{A.11}$$

in $B_{\epsilon^{10}}(x_0)$, and

$$\begin{aligned}&(\mathbf{D}^2 h_\epsilon)_{ab} = O(1), \quad (\mathbf{D}^2 h_\epsilon)_{3a} = O(1), \quad (\mathbf{D}^2 h_\epsilon)_{4a} = O(1), \quad a, b = 1, 2,\\ &(\mathbf{D}^2 h_\epsilon)_{33} = O(1), \quad (\mathbf{D}^2 h_\epsilon)_{44} = O(1), \quad (\mathbf{D}^2 h_\epsilon)_{34} = -\Omega^2\epsilon^{-1} + O(1)\end{aligned} \tag{A.12}$$

in $B_{\epsilon^{10}}(x_0)$, where $O(1)$ denotes various functions on $B_{\epsilon^{10}}(x_0)$ with absolute value bounded by constants that depends only on A. Thus

$$\mathbf{D}^\alpha h_\epsilon(x_0)\mathbf{D}^\beta h_\epsilon(x_0)(\mathbf{D}_\alpha h_\epsilon \mathbf{D}_\beta h_\epsilon - \epsilon \mathbf{D}_\alpha \mathbf{D}_\beta h_\epsilon)(x_0) = 2 + \epsilon O(1).$$

This proves (A.3) if ϵ_1 is sufficiently small. Similarly, if $X = X^\alpha e_\alpha$ then, with $\mu = \epsilon_1^{-1/2}$ we compute

$$\begin{aligned}&X^\alpha X^\beta(\mu \mathbf{g}_{\alpha\beta} - \mathbf{D}_\alpha \mathbf{D}_\beta h_\epsilon)(x_0) + \epsilon^{-2}|X^\alpha \mathbf{D}_\alpha h_\epsilon(x_0)|^2\\ &= \mu((X^1)^2 + (X^2)^2) + 2(\epsilon^{-1} - \mu)X^3X^4 + \epsilon^{-2}(X^3 - X^4)^2 + O(1)\sum_{\alpha=1}^{4}(X^\alpha)^2\\ &\geq (\mu/2)((X^1)^2 + (X^2)^2) + (\epsilon^{-1}/2)((X^3)^2 + (X^4)^2),\end{aligned}$$

provided that ϵ_1 is sufficiently small. This completes the proof of (A.4).

It follows from the Carleman estimates in Lemmas A.2 and A.3 that there is $\epsilon = \epsilon(A) \in (0, c)$ (where c is the constant in Lemma 4.4) and a constant $\widetilde{C} = \widetilde{C}(A) \geq 1$ such that

$$\begin{aligned}&\lambda \|e^{-\lambda f_\epsilon}\phi\|_{L^2} + \|e^{-\lambda f_\epsilon}|\partial\phi|\,\|_{L^2} \leq \widetilde{C}\lambda^{-1/2}\|e^{-\lambda f_\epsilon}\,\Box_{\mathbf{g}}\phi\|_{L^2};\\ &\|e^{-\lambda f_\epsilon}\phi\|_{L^2} \leq \widetilde{C}\lambda^{-1}\|e^{-\lambda f_\epsilon}\underline{L}(\phi)\|_{L^2},\end{aligned} \tag{A.13}$$

for any $\phi \in C_0^\infty(B_{\epsilon^{10}}(x_0))$ and any $\lambda \geq \widetilde{C}$, where $f_\epsilon = \ln(h_\epsilon + e_\epsilon)$. Let $\eta : \mathbb{R} \to [0, 1]$ denote a smooth function supported in $[1/2, \infty)$ and equal to 1 in $[3/4, \infty)$. For $\delta \in (0, 1]$, $i = 1, \ldots, I$, $j = 1, \ldots J$ we define,

$$\begin{aligned}&G_i^{\delta,\epsilon} = G_i \cdot \mathbf{1}_{I_c^{+-}} \cdot \eta(-u\underline{u}/\delta) \cdot \big(1 - \eta(N^{x_0}/\epsilon^{20})\big) = G_i \cdot \widetilde{\eta}_{\delta,\epsilon}\\ &H_j^{\delta,\epsilon} = H_j \cdot \mathbf{1}_{I_c^{+-}} \cdot \eta(-u\underline{u}/\delta) \cdot \big(1 - \eta(N^{x_0}/\epsilon^{20})\big) = H_j \cdot \widetilde{\eta}_{\delta,\epsilon}.\end{aligned} \tag{A.14}$$

Clearly, $G_i^{\delta,\epsilon}, H_j^{\delta,\epsilon} \in C_0^\infty(B_{\epsilon^{10}}(x_0) \cap \mathbf{E})$. We would like to apply the inequalities in (A.13) to the functions $G_i^{\delta,\epsilon}, H_j^{\delta,\epsilon}$, and then let $\delta \to 0$ and $\lambda \to \infty$ (in this order).

Using the definition (A.14), we have

$$\begin{aligned}&\square_{\mathbf{g}} G_i^{\delta,\epsilon} = \widetilde{\eta}_{\delta,\epsilon} \cdot \square_{\mathbf{g}} G_i + 2\mathbf{D}_\alpha G_i \cdot \mathbf{D}^\alpha \widetilde{\eta}_{\delta,\epsilon} + G_i \cdot \square_{\mathbf{g}} \widetilde{\eta}_{\delta,\epsilon};\\ &\underline{L}(H_j^{\delta,\epsilon}) = \widetilde{\eta}_{\delta,\epsilon} \cdot \underline{L}(H_j) + H_j \cdot \underline{L}(\widetilde{\eta}_{\delta,\epsilon}).\end{aligned}$$

Using the Carleman inequalities (A.13), for any $i = 1, \dots, I$, $j = 1, \dots, J$ we have

$$\begin{aligned}&\lambda \cdot \|e^{-\lambda f_\epsilon} \cdot \widetilde{\eta}_{\delta,\epsilon} G_i\|_{L^2} + \|e^{-\lambda f_\epsilon} \cdot \widetilde{\eta}_{\delta,\epsilon} |\partial^1 G_i|\,\|_{L^2} \leq \widetilde{C} \lambda^{-1/2} \cdot \|e^{-\lambda f_\epsilon} \cdot \widetilde{\eta}_{\delta,\epsilon} \square_{\mathbf{g}} G_i\|_{L^2}\\ &+ \widetilde{C} \Big[\|e^{-\lambda f_\epsilon} \cdot \mathbf{D}_\alpha G_i \mathbf{D}^\alpha \widetilde{\eta}_{\delta,\epsilon}\|_{L^2} + \|e^{-\lambda f_\epsilon} \cdot G_i (|\square_{\mathbf{g}} \widetilde{\eta}_{\delta,\epsilon}| + |\partial^1 \widetilde{\eta}_{\delta,\epsilon}|)\|_{L^2} \Big]\end{aligned} \tag{A.15}$$

and

$$\|e^{-\lambda f_\epsilon} \cdot \widetilde{\eta}_{\delta,\epsilon} H_j\|_{L^2} \leq \widetilde{C} \lambda^{-1} \|e^{-\lambda f_\epsilon} \cdot \widetilde{\eta}_{\delta,\epsilon} \underline{L}(H_j)\|_{L^2} + \widetilde{C} \lambda^{-1} \|e^{-\lambda f_\epsilon} \cdot H_j \underline{L}(\widetilde{\eta}_{\delta,\epsilon})\|_{L^2}, \tag{A.16}$$

for any $\lambda \geq \widetilde{C}$. Using the main identities (4.12), in $B_{\epsilon^{10}}(x_0)$ we estimate pointwise

$$\begin{aligned}&|\square_{\mathbf{g}} G_i| \leq M \sum_{l=1}^{I} \big(|\partial^1 G_l| + |G_l|\big) + M \sum_{m=1}^{J} |H_j|,\\ &|\underline{L}(H_j)| \leq M \sum_{l=1}^{I} \big(|\partial^1 G_l| + |G_l|\big) + M \sum_{m=1}^{J} |H_j|,\end{aligned} \tag{A.17}$$

for some large constant M. We add inequalities (A.15) and (A.16) over i, j. The key observation is that, in view of (A.17), the first terms in the right-hand sides of (A.15) and (A.16) can be absorbed into the left-hand sides for λ sufficiently large. Thus, for any λ sufficiently large and $\delta \in (0, 1]$,

$$\begin{aligned}&\lambda \sum_{i=1}^{I} \|e^{-\lambda f_\epsilon} \cdot \widetilde{\eta}_{\delta,\epsilon} G_i\|_{L^2} + \sum_{j=1}^{J} \|e^{-\lambda f_\epsilon} \cdot \widetilde{\eta}_{\delta,\epsilon} H_j\|_{L^2} \leq \widetilde{C} \lambda^{-1} \sum_{j=1}^{J} \|e^{-\lambda f_\epsilon} \cdot H_j |\partial \widetilde{\eta}_{\delta,\epsilon}|\|_{L^2}\\ &+ \widetilde{C} \sum_{i=1}^{I} \Big[\|e^{-\lambda f_\epsilon} \cdot \mathbf{D}_\alpha G_i \mathbf{D}^\alpha \widetilde{\eta}_{\delta,\epsilon}\|_{L^2} + \|e^{-\lambda f_\epsilon} \cdot G_i (|\square_{\mathbf{g}} \widetilde{\eta}_{\delta,\epsilon}| + |\partial \widetilde{\eta}_{\delta,\epsilon}|)\|_{L^2} \Big].\end{aligned} \tag{A.18}$$

We let now $\delta \to 0$ and $\lambda \to \infty$, as in [10, Section 6], to conclude that $\mathbf{1}_{B_{\epsilon^{40}}(x_0) \cap I^{+-}} G_i = 0$ and $\mathbf{1}_{B_{\epsilon^{40}}(x_0) \cap I^{+-}} H_j = 0$. The main ingredient needed for this limiting procedure is the inequality

$$\inf_{B_{\epsilon^{40}}(x_0) \cap I_c^{+-}} e^{-\lambda f_\epsilon} \geq e^{\lambda / \widetilde{C}} \sup_{\{x \in B_{\epsilon^{10}}(x_0) \cap I_c^{+-} : N^{x_0} \geq \epsilon^{20}/2\}} e^{-\lambda f_\epsilon},$$

which follows easily from the definition (A.10). The lemma follows. □

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