On the decomposition of Global Conformal Invariants II

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Abstract

This paper is a continuation of [2], where we complete our partial proof of the Deser-Schwimmer conjecture on the structure of "global conformal invariants". Our theorem deals with such invariants $P(g^n)$ that locally depend only on the curvature tensor R_{ijkl} (without covariant derivatives).

In [2] we developed a powerful tool, the "super divergence formula" which applies to any Riemannian operator that always integrates to zero on compact manifolds. In particular, it applies to the operator $I_{g^n}(\phi)$ that measures the "non-conformally invariant part" of $P(g^n)$. This paper resolves the problem of using this information we have obtained on the structure of $I_{g^n}(\phi)$ to understand the structure of $P(g^n)$.

1 Introduction

We briefly recall the open problem that this paper and [2] address and the theorem that we will be completing here. Our objects of study are *scalar Riemannian invariants* $P(g^n)$ of a Riemannian manifold (M^n, g^n). These are polynomials in the components of the tensors $R_{ijkl}, \dots, \nabla^m_{r_1\dots r_m} R_{ijkl}, \dots$ and g^{ij} (or, even more generally, in the variables $\partial^k_{t_1\dots t_k} g_{ij}$, $det(g)^{-1}$), that are independent of the coordinate system in which they are expressed, and also have a weight W, meaning that under a re-scaling $g^n \to t^2 g^n$ they transform by $P(t^2 g^n) = t^W P(g^n)$, $t \in \mathbb{R}_+$. It is a classical result that such invariants are linear combinations

$$P(g^n) = \Sigma_{l \in L} a_l C^l(g^n) \tag{1}$$

of complete contractions in the form:

$$contr(\nabla^{m_1}_{r_1\dots r_{m_1}} R_{ijkl} \otimes \dots \otimes \nabla^{m_s}_{t_1\dots t_{m_s}} R_{i'j'k'l'}) \tag{2}$$

each with weight W. We fix an *even* dimension n once and for all, and we restrict attention to local scalar invariants of weight $-n$. Due to the transformation of the volume form $dV_{e^{2\phi(x)}g^n} = e^{n\phi(x)} dV_{g^n}$ under general conformal re-scalings $\hat{g}^n \to e^{2\phi(x)} g^n$, it follows that if $P(g^n)$ has weight $-n$ then the quantity $\int_{M^n} P(g^n) dV_{g^n}$ is scale-invariant for any compact orientable Riemannian (M^n, g^n).

The problem we are addressing is to find all Riemannian scalar invariants of weight $-n$ for which the integral $\int_{M^n} P(g^n)dV_{g^n}$ is invariant under conformal re-scalings $\hat{g}^n = e^{2\phi(x)}g^n$ for any compact manifold (M^n, g^n) and any $\phi \in C^\infty(M^n)$. In other words, we are assuming that for any (M^n, g^n) and $\phi \in C^\infty(M^n)$ we must have:

$$\int_{M^n} P(g^n)dV_{g^n} = \int_{M^n} P(\hat{g}^n)dV_{\hat{g}^n} \tag{3}$$

Deser and Schwimmer, two physicists, conjectured the following in [10]:

Conjecture 1 (Deser-Schwimmer) *Suppose we have a Riemannian scalar $S(g^n)$ of weight $-n$ for some even n. Suppose that for any compact manifold (M^n, g^n) the quantity*

$$\int_{M^n} S(g^n)dV_{g^n} \tag{4}$$

is invariant under any conformal change of metric $\hat{g}^n(x) = e^{2\phi(x)}g^n(x)$. Then $P(g^n)$ must be a linear combination of three"obvious candidates", namely:

$$S(g^n) = W(g^n) + div_iT_i(g^n) + c \cdot \text{Pfaff}(R_{ijkl}) \tag{5}$$

1. *$W(g^n)$ is a scalar conformal invariant of weight $-n$, ie it satisfies $W(e^{2\phi(x)}g^n) = e^{-n\phi(x)}W(g^n)$ for every $\phi \in C^\infty(M^n)$ and every $x \in M^n$.*

2. *$T_i(g^n)$ is a Riemannian vector field of weight $-n+1$. (Since for any compact M^n we have $\int_{M^n} div_iT_i(g^n)dV_{g^n} = 0$.)*

3. *Pfaff(R_{ijkl}) stands for the Pfaffian of the curvature R_{ijkl}. (Since for any compact Riemannian (M^n, g^n) $\int_{M^n} \text{Pfaff}(R_{ijkl})dV_{g^n} = \frac{2^n\pi^{\frac{n}{2}}(\frac{n}{2}-1)!}{2(n-1)!}\chi(M^n)$.)*

In this paper we complete our partial confirmation of this conjecture. We restrict our attention to Riemannian scalars $P(g^n)$ that are linear combinations

$$\Sigma_{l\in L}a_lC^l(g^n) \tag{6}$$

of complete contractions of weight $-n$, each $C^l(g^n)$ in the form:

$$contr(R_{i_1j_1k_1l_1} \otimes \cdots \otimes R_{i_{\frac{n}{2}}j_{\frac{n}{2}}k_{\frac{n}{2}}l_{\frac{n}{2}}}) \tag{7}$$

(since we are not allowing derivatives on the factors R_{ijkl}, the weight restriction forces each complete contraction to have $\frac{n}{2}$ factors). The main theorem that we show in [2] and in the present paper is:

Theorem 1 *Let us suppose that $P(g^n)$ is in the form (6), where each $C^l(g^n)$ is in the form (7), with $r = \frac{n}{2}$ factors. We also assume that (3) holds for any Riemannian (M^n, g^n) and $\phi \in C^\infty(M^n)$.*

Then, there exists a a scalar conformal invariant $W(g^n)$ of weight $-n$ that locally depends only on the Weyl tensor, and also a constant c so that:

$$S(g^n) = W(g^n) + c \cdot \text{Pfaff}(R_{ijkl}) \tag{8}$$

where $\text{Pfaff}(R_{ijkl})$ *stands for the Pfaffian of the curvature* R_{ijkl}.

We will recall two related results that were proven by entirely different methods. In [17] Gilkey considered the problem of finding all scalar invariants $P(g^n)$ of weight $-n$ for which $\int_{M^n} P(g^n)dV_{g^n}$ is constant for a given compact orientable M^n and *any* Riemannian metric g^n over M^n. He then showed that:

Theorem 2 (Gilkey) *Under the above assumptions, we have that $P(g^n)$ can be written as:*

$$P(g^n) = div_i T_i(g^n) + c \cdot \text{Pfaff}(R_{ijkl}) \tag{9}$$

where $T_i(g^n)$ is an intrinsic vector field of weight $-n+1$ and $\text{Pfaff}(R_{ijkl})$ *stands for the Pfaffian of the curvature tensor.*

(see also [24] for an earlier form of this result). Extending the methods in [17], Branson, Gilkey and Pohjanpelto showed in [5] that:

Theorem 3 (Branson-Gilkey-Pohjanpelto) *Consider any local Riemannian invariant $P(g^n)$ of weight $-n$, with the property that for any manifold M^n and any locally conformally flat metric h^n, $\int_{M^n} P(h^n)dV_{h^n}$ is invariant under conformal re-scalings $\hat{h}^n = e^{2\phi(x)}h^n$ of the metric h^n. It then follows that in the locally conformally flat metric h^n (for which the Weyl tensor vanishes), we can write out:*

$$P(h^n) = div_i T_i(h^n) + c \cdot \text{Pfaff}(R_{ijkl}) \tag{10}$$

where $T_i(h^n)$ is a vector field of weight $-n+1$ and $\text{Pfaff}(R_{ijkl})$ *stands for the Pfaffian of the curvature tensor.*

We have explained in [2] how resolving the whole of the Deser-Schwimmer conjecture would have implications regarding the structure of the so-called Q-curvature, and also for the study of conformally compact Einstein manifolds, in particular regarding the notions of the re-normalized volume and the conformal anomaly, see also [1], [9], [18], [21], [20],[23]. Here, we briefly recall the definition of Q-curvature.

Q-curvature is a Riemannian scalar invariant $Q^n(g^n)$ constructed by Branson for each even dimension n (see [4]). In dimension 2 it is just the scalar curvature ($Q^2(g^2) = R$) and in dimension 4 (where it has been extensively studied), it is in the form:

$$Q^4(g^4) = \frac{1}{12}\big(- \Delta R + \frac{1}{4}R^2 - |E|^2\big) \tag{11}$$

where R is the scalar curvature and E is the traceless Ricci tensor.

In dimension n $Q^n(g^n)$ has weight $-n$. Its two main properties are that $\int_{M^n} Q^n(g^n)dV_{g^n}$ is invariant under conformal changes of g^n and that under the re-scaling $g^n \to e^{2\phi(x)}g^n$, $Q^n(g^n)$ enjoys the transformation law:

$$Q^n(e^{2\phi(x)}g^n)(x) = e^{-n\phi(x)}[Q^n(g^n) + P^{\frac{n}{2}}_{g^n}(\phi)](x) \tag{12}$$

where $P^{\frac{n}{2}}_{g^n}(\phi)$ is a *conformally co-variant differential operator*, originally constructed in [19]. Conformal co-variance means that its symbol has a nice transformation law under the conformal re-scaling $\hat{g}^n = e^{2\phi(x)}g^n$, namely for every g^n, $\phi, \psi \in C^\infty(M^n)$:

$$P^{\frac{n}{2}}_{e^{2\psi(x)}g^n}(\phi) = e^{-n\psi(x)}P^{\frac{n}{2}}_{g^n}(\phi) \tag{13}$$

The above transformation law has played an important role in the analysis surrounding Q-curvature (see [7], [6] for example). Moreover, the particular form of $Q^4(g^4)$ and its relation to the Chern-Gauss-Bonnet integrand has proven to be a valuable tool in geometric and topological applications of Q-curvature in dimension 4, see [8], [25]. Therefore, understanding of the structure of Q-curvature in high dimensions would raise the question whether the powerful techniques employed in the study of Q-curvature in dimension 4 can be extended to higher dimensions.

2 Formulas and an outline of the proof.

Throughout this paper we will be employing all the notational and terminological conventions from [2]. We will also be heavily using Theorem 2 in that paper and its two corollaries regarding identities that hold "formally" or "by substitution", see also [3], [13], [26].

We recall that $P(g^n)$ satisfies (3). In [2] we defined an operator $I_{g^n}(\phi)$ as:

$$I_{g^n}(\phi) = e^{n\phi(x)}P(e^{2\phi(x)}g^n) - P(g^n) \tag{14}$$

which has weight $-n$ and the fundamental property that:

$$\int_{M^n} I_{g^n}(\phi)dV_{g^n} = 0 \tag{15}$$

for every compact Riemannian (M^n, g^n).

As our tool for this paper will be the super divergence formula for $I_{g^n}(\phi)$, it is necessary to write out $P(g^n)$ in such a way so that we can "recover" the non-conformally invariant part of $P(g^n)$ from the expression of $I_{g^n}(\phi)$. As an illustration of the difficulty that we are forced to address, we suppose that we write out $P(g^n)$ as a linear combination of contractions in the form (7). But then, given the transformation law for the curvature tensor, it is not obvious how to reconstruct $P(g^n)$ if we are given $I_{g^n}(\phi)$.

In order to overcome this difficulty, we recall the Schouten tensor as a trace-adjustment of Ricci curvature:

$$P_{\alpha\beta} = \frac{1}{n-2}[Ric_{\alpha\beta} - \frac{R}{2(n-1)}g^n_{\alpha\beta}] \tag{16}$$

Where $Ric_{\alpha\beta}$ stands for Ricci curvature and R stands for scalar curvature. We then have the well-known decomposition of the curvature tensor:

$$R_{ijkl} = W_{ijkl} + [P_{jk}g^n_{il} + P_{il}g^n_{jk} - P_{jl}g^n_{ik} - P_{ik}g^n_{jl}] \tag{17}$$

The Weyl tensor is trace-free and conformally invariant, ie for $\hat{g}^n = e^{2\phi}g^n$:

$$W^{\hat{g}^n}_{ijkl} = e^{2\phi(x)}W^{g^n}_{ijkl} \tag{18}$$

While the Schouten tensor has the following transformation law:

$$P^{\hat{g}^n}_{\alpha\beta} = P^{g^n}_{\alpha\beta} - \phi_{\alpha\beta} + \phi_\alpha\phi_\beta - \frac{1}{2}\phi^k\phi_k g^n_{\alpha\beta} \tag{19}$$

In view of our assumption for Theorem 1 and equation (17), we may now write $P(g^n)$ in the form:

$$P(g^n) = \Sigma_{l\in L} a_l C^l(g^n) \tag{20}$$

where each complete contraction $C^l(g^n)$ is in the from:

$$contr(W_{i_1j_1k_1l_1} \otimes \dots \otimes W_{i_Aj_Ak_Al_A} \otimes P_{a_1b_1} \otimes \dots \otimes P_{a_Bb_B}) \tag{21}$$

Because of the weight restriction, we see that $A + B = \frac{n}{2}$.

Let us break up the index set L into subsets $L^{\mu,\nu}$ as follows: $l \in L^{\mu,\nu}$ if and only if $C^l(g^n)$ is in the above form and $A = \mu, B = \nu$.

We then notice that the linear combination:

$$P^1(g^n) = \Sigma_{l\in L^{\frac{n}{2},0}} a_l C^l(g^n)$$

is a scalar conformal invariant of weight $-n$. Hence, in view of the claim of our Theorem 1, we may subtract it off, and we are left with considering the case where $P(g^n)$ is a linear combination:

$$P(g^n) = \Sigma_{l\in L} a_l C^l(g^n)$$

where each complete contraction $C^l(g^n)$ is in the form (21) with $B \geq 1$.

We then have the main theorem of this paper:

Theorem 4 *Suppose we are given a* $P(g^n)$ *which is a linear combination of complete contractions of weight* $-n$*, each in the form (21) with* $B \geq 1$ *and* $P(g^n)$ *satisfies (3). Suppose we know the coefficient of the complete contraction* $(P_a^a)^{\frac{n}{2}}$ *in* $P(g^n)$*.*

Then there can be at most one linear combination $P(g^n)$ *of complete contractions in the form (21) with* $B \geq 1$ *for which the condition (3) holds.*

If we can show the above, our Theorem 1 will follow. In order to see this, observe that for each even dimension n, we have that $\mathrm{Pfaff}(R_{ijkl})$ cannot be a linear combination of complete contractions depending only on the Weyl curvature: If for some n that were the case, we would have that for the n-sphere S^n with the standard locally conformally flat metric $\int_{S^n} \mathrm{Pfaff}(R_{ijkl}) dV_{g^n} = 0$, which is absurd by the Chern-Gauss-Bonnet Theorem.

Thus, if we write out $\mathrm{Pfaff}(R_{ijkl})$ as a linear combination of complete contractions in the form (21) and define $\overline{\mathrm{Pfaff}}(R_{ijkl})$ to stand for the sublinear combination of the complete contractions in $\mathrm{Pfaff}(R_{ijkl})$ with $B \geq 1$, we will deduce that for some constant C, $P(g^n)$ in Theorem 4 can be written as:

$$P(g^n) = C \cdot \overline{\mathrm{Pfaff}}(R_{ijkl}) \tag{22}$$

This implies our main theorem. □

We will prove Theorem 4 by the following two Lemmas:

Lemma 1 *Given the coefficient of the complete contraction* $(P_a^a)^{\frac{n}{2}}$*, there can be at most one sublinear combination of complete contractions* $C^l(g^n)$ *of the form (21) in* $P(g^n)$ *with* $A = 0, B = \frac{n}{2}$ *so that (3) holds.*

Lemma 2 *Given an integer* $1 \leq A_1 \leq \frac{n}{2} - 1$*, and given the sublinear combination of the complete contractions* $C^l(g^n)$ *in* $P(g^n)$ *with* $A < A_1$*, then there can be at most one sublinear combination of complete contractions* $C^l(g^n)$ *of the form (21) in* $P(g^n)$ *with* $A = A_1$ *so that (3) holds.*

It is clear that if we can prove the above two Lemmas, then by induction Theorem 4 will follow. In the rest of the paper we give the proof of these Lemmas.

Our main tool in the proof will be the super divergence formula and the shadow divergence formula used on the operator $I_{g^n}(\phi)$.

A disclaimer on our use of these formulas is in order. We will no longer be needing the polarized form $I^Z_{g^n}(\psi_1, \ldots, \psi_Z)$ of $I^Z_{g^n}(\phi)$. We will be referring to the super divergence formula of $I^Z_{g^n}(\phi)$, and we will mean the formula that arises from $supdiv[I^Z_{g^n}(\psi_1, \ldots, \psi_Z)]$ by setting $\psi_1 = \cdots = \psi_Z = \phi$ and dividing by $Z!$. The same will apply when we refer to the shadow divergence formula of $I^Z_{g^n}(\phi)$.

We must also recall a few more simple facts from [2]. We recall that $I^Z_{g^n}(\phi)$ is taken to be a linear combination of complete contractions in the form:

$$contr(\nabla^{m_1}_{r_1\dots r_{m_1}}R_{ijkl}\otimes\dots\otimes\nabla^{m_s}_{t_1\dots t_{m_s}}R_{ijkl}\otimes \\ \nabla^{\nu_1}_{a_1\dots a_{\nu_1}}\phi\otimes\dots\otimes\nabla^{\nu_Z}_{b_1\dots b_{\nu_Z}}\phi) \tag{23}$$

We also recall that in the context of the iterative integrations by parts, the $\vec{\xi}$-contractions that we generically encounter are in the form:

$$contr(\nabla^{m_1}_{r_1\dots r_{m_1}}R_{i_1j_1k_1l_1}\otimes\dots\otimes\nabla^{m_s}_{v_1\dots v_{m_s}}R_{i_sj_sk_sl_s}\otimes\nabla^{\nu_1}_{\chi_1\dots\chi_{\nu_1}}\phi\otimes\dots\otimes\nabla^{\nu_Z}_{\omega_1\dots\omega_{\nu_Z}}\phi \\ \otimes\vec{\xi}\otimes\dots\otimes\vec{\xi}\otimes S[\nabla^{w_1}_{u_1\dots u_{w_1}}\vec{\xi}]\otimes\dots\otimes S[\nabla^{w_l}_{q_1\dots q_{w_l}}\vec{\xi}]) \tag{24}$$

where the factors $\nabla^m R_{ijkl}$ are allowed to have internal contractions among the indices i,j,k,l.

Upon occasion, we will be writing those complete contractions as linear combinations of complete contractions in the forms:

$$contr(\nabla^{m_1}_{r_1\dots r_{m_1}}R_{ijkl}\otimes\dots\otimes\nabla^{m_s}_{t_1\dots t_{m_s}}R_{ijkl}\otimes \\ S\nabla^{\nu_1}_{a_1\dots a_{\nu_1}}\phi\otimes\dots\otimes S\nabla^{\nu_Z}_{b_1\dots b_{\nu_Z}}\phi) \tag{25}$$

$$contr(\nabla^{m_1}_{r_1\dots r_{m_1}}R_{i_1j_1k_1l_1}\otimes\dots\otimes\nabla^{m_s}_{v_1\dots v_{m_s}}R_{i_sj_sk_sl_s}\otimes\otimes S\nabla^{\nu_1}_{\chi_1\dots\chi_{\nu_1}}\phi\otimes\dots\otimes \\ S\nabla^{\nu_Z}_{\omega_1\dots\omega_{\nu_Z}}\phi\otimes\vec{\xi}\otimes\dots\otimes\vec{\xi}\otimes S[\nabla^{w_1}_{u_1\dots u_{w_1}}\vec{\xi}]\otimes\dots\otimes S[\nabla^{w_l}_{q_1\dots q_{w_l}}\vec{\xi}]) \tag{26}$$

One immediately sees that we can write each complete contraction in the form (23) or (24) as a linear combination of contractions in the forms (25) or (26) by repeated use of the identity:

$$[\nabla_i\nabla_j-\nabla_j\nabla_i]X_k=R_{ijkl}X^l \tag{27}$$

We must also recall the transformation law of the curvature tensor, along with that of the Levi-Civita connection, under conformal re-scalings $\hat{g}^n=e^{2\phi(x)}g^n$:

$$R^{\hat{g}^n}_{ijkl}=e^{2\phi(x)}[R^{g^n}_{ijkl}+\phi_{il}g_{jk}+\phi_{jk}g_{il}-\phi_{ik}g_{jl}-\phi_{jl}g_{ik}+\phi_i\phi_kg_{jl}+\phi_j\phi_lg_{ik} \\ -\phi_i\phi_lg_{jk}-\phi_j\phi_kg_{il}+|\nabla\phi|^2g_{il}g_{jk}-|\nabla\phi|^2g_{ik}g_{lj}] \tag{28}$$

$$\nabla^{\hat{g}^n}_k\eta_l=\nabla^{g^n}_k\eta_l-\phi_k\eta_l-\phi_l\eta_k+\phi^s\eta_sg^n_{kl} \tag{29}$$

Next, we will prove certain Lemmas that will be useful throughout this paper.

2.1 Useful Lemmas.

Our first Lemma is the following:

Lemma 3 *Suppose we are given a collection of complete contractions $C^k_{g^n}(\phi)$, $k \in K$ of weight $-n$ and in the form (25) or a collection of complete contractions $C^k_{g^n}(\phi, \vec{\xi})$, $k \in K$, each in the form (26). Suppose that the identities, respectively:*

$$\Sigma_{k\in K} a_k C^k_{g^n}(\phi) = 0 \tag{30}$$

$$\Sigma_{k\in K} a_k C^k_{g^n}(\phi, \vec{\xi}) = 0 \tag{31}$$

hold for every Riemannian manifold (M^n, g^n) at any point x_0 and for any function ϕ defined around x_0, and in the second case for any vector $\vec{\xi} \in \mathbb{R}^n$. We define subsets $K^{(r_1,\dots,r_Z)}$ of the index set K as follows: $k \in K^{(r_1,\dots,r_Z)}$ if and only if $C^k_{g^n}(\phi)$, which is in the form (25), satisfies $\nu_1 = r_1, \dots, \nu_Z = r_Z$, where the values $\nu_1, \dots, \nu_Z$ are taken in decreasing rearrangement.

Then, for any subset $K^{(r_1,\dots,r_Z)} \subset K$, we will have, respectively:

$$\Sigma_{k\in K^{(r_1,\dots,r_Z)}} a_k C^k_{g^n}(\phi) = 0 \tag{32}$$

$$\Sigma_{k\in K^{(r_1,\dots,r_Z)}} a_k C^k_{g^n}(\phi, \vec{\xi}) = 0 \tag{33}$$

for any Riemannian manifold (M^n, g^n) at any point x_0 and for any function ϕ defined around x_0, and in the second case for any vector $\vec{\xi} \in \mathbb{R}^n$.

Proof: We only have to observe that the relations (30) and (31) hold formally, where we regard the tensors $S\nabla^\nu_{r_1\dots r_\nu}\phi$ as symmetric p-tensors $\Omega_{r_1\dots r_\nu}$. On the other hand, the values $\nu_1, \dots, \nu_Z$ remain invariant under the permutation relations of Definitions 7 and 8 in [2]. Hence, we have our Lemma. □

Our second Lemma will be the following:

Lemma 4 *Let us suppose we are given complete contractions $C^k_{g^n}(\phi)$ in the form (23), and that the identity:*

$$\Sigma_{k\in K} a_k C^k_{g^n}(\phi)(x_0) = 0$$

holds on any Riemannian manifold (M^n, g^n) and for any function ϕ around x_0. Let us suppose that the minimum length among the complete contractions $\{C^k_{g^n}(\phi)\}_{k\in K}$ is L. Then let us define the subset $K^\sharp \subset K$ as follows: $k \in K^\sharp$ if and only if $C^k_{g^n}(\phi)$ which is in the form (23), has length L and also has no internal contractions. We then have that:

$$\Sigma_{k\in K^\sharp} a_k C^k_{g^n}(\phi) = 0 \tag{34}$$

modulo complete contractions of length $\geq L + 1$.

Proof: Let us begin by defining the set $K_1 \subset K$ as follows: $k \in K_1$ if and only if $C^k_{g^n}(\phi)$ has length L. Obviously, $K^\sharp \subset K_1$.

Now, we want to apply Theorem 2 in [2]. For each complete contraction $C^k_{g^n}(\phi), k \in K_1$, we consider its *linearization* $linC^l(R,\phi)$. Then, by the Lemma hypothesis and Theorem 2 in [2], we have that the equation:

$$\Sigma_{k\in K^\sharp} a_k linC^k(R,\phi) + \Sigma_{k\in K_1\setminus K^\sharp} a_k linC^l(R,\phi) = 0 \tag{35}$$

will hold formally. But then notice the following: For any linearized complete contraction $linC(R,\phi)$, the number of internal contractions remains unaltered under any of the linearized permutation identities. Hence, (35) implies that:

$$\Sigma_{k\in K^\sharp} a_k linC^k(R,\phi) = 0$$

formally. But then, as in the proof of the corollaries of Theorem 2 in [2], we have that:

$$\Sigma_{k\in K^\sharp} a_k C^k_{g^n}(\phi) = 0$$

modulo complete contractions of length $\geq L+1$. □

3 The easier step: Proof of Lemma 1.

Consider any complete contraction $C^l(g^n)$ in the form (21) with $A=0$. Let us denote by $R[C^l(g^n)]$ the number of factors P^a_a in $C^l(g^n)$. Also, let $L^{0,\frac{n}{2},\lambda}$ stand for the subset of L which is defined as follows: $l \in L^{0,\frac{n}{2},\lambda}$ if and only if $l \in L^{0,\frac{n}{2}}$ and $R[C^l(g^n)] = \lambda$.

We will show Lemma 1 by an inductive statement. We assume that for some $T \geq 0$, we have determined the sublinear combinations $\Sigma_{l\in L^{0,\frac{n}{2},\lambda}} a_l C^l(g^n)$, for each $\lambda \geq T+1$. We will then show that we can determine the sublinear combination $\Sigma_{l\in L^{0,\frac{n}{2},T}} a_l C^l(g^n)$. If we can prove this inductive step, then it is obvious that our Lemma will follow.

In order to prove the above, we consider $I^{\frac{n}{2}}_{g^n}(\phi)$. For any $C^l(g^n)$ with $l \in L^{0,\frac{n}{2}}$, we define $C^l_{g^n}(\phi)$ to be the complete contraction which is obtained from $C^l(g^n)$ by substituting each factor P_{ab} by $-\nabla^2_{ab}\phi$.

By virtue of (19) and the definition of $I^{\frac{n}{2}}_{g^n}(\phi)$ we have that:

$$I^{\frac{n}{2}}_{g^n}(\phi) = \Sigma_{l\in L^{0,\frac{n}{2}}} a_l C^l_{g^n}(\phi)$$

modulo complete contractions of length $\geq \frac{n}{2}+1$. In particular, each $C^l(g^n)$ with $l \in L^{A,B}$, $A \geq 1$ will not contribute to the above.

So the problem is reduced to determining the sublinear combination $\Sigma_{l\in L^{0,\frac{n}{2},T}} a_l C^l_{g^n}(\phi)$ of complete contractions $C^l_{g^n}(\phi)$ with T factors $\Delta\phi$ from the sublinear combination $\Sigma_{s=T+1}^{\frac{n}{2}}\Sigma_{l\in L^{0,\frac{n}{2},T}} a_l C^l_{g^n}(\phi)$ of complete contractions $C^l_{g^n}(\phi)$ with more than T factors $\Delta\phi$.

We will use the formula $supdiv[I^{\frac{n}{2}}_{g^n}(\phi)]$. Let us make a definition:

Consider any complete contraction $C^l_{g^n}(\phi)$, $l \in L^{0,\frac{n}{2},T}$. It will be in the form:

$$contr(\nabla^2_{a_1b_1}\phi\otimes\dots\otimes\nabla^2_{a_{\frac{n}{2}-T}b_{\frac{n}{2}-T}}\phi\otimes\Delta\phi\otimes\dots\otimes\Delta\phi)$$

where none of the factors $\nabla^2_{a_ib_i}\phi$ is in the form $\Delta\phi$.

We consider the complete contraction $C^{l,D}_{g^n}(\phi)$:

$$contr(\nabla^{i_1\dots i_T}[\nabla^2_{a_1b_1}\phi\otimes\dots\otimes\nabla^2_{a_{\frac{n}{2}-T}b_{\frac{n}{2}-T}}\phi]\otimes\nabla_{i_1}\phi\otimes\dots\otimes\nabla_{i_T}\phi)$$

We write out $C^{l,D}_{g^n}(\phi)$ as a linear combination $\Sigma_{r\in R^l} a_r C^r_{g^n}(\phi)$, where each $C^r_{g^n}(\phi)$ is in the form:

$$contr(\nabla^{m_1}_{r_1\dots r_{m_1}}\phi\otimes\dots\otimes\nabla^{m_{\frac{n}{2}-T}}_{w_1\dots w_{m_{\frac{n}{2}-T}}}\phi\otimes\nabla_{i_1}\phi\otimes\dots\otimes\nabla_{i_T}\phi)$$

where each $m_i\geq 2$ and each index i_s contracts against an index in a factor $\nabla^{m_e}\phi$. For each such complete contraction $C^r_{g^n}(\phi)$, we define $SC^r_{g^n}(\phi)$ to be:

$$contr(S\nabla^{m_1}_{r_1\dots r_{m_1}}\phi\otimes\dots\otimes S\nabla^{m_{\frac{n}{2}-T}}_{w_1\dots w_{m_{\frac{n}{2}-T}}}\phi\otimes\nabla_{i_1}\phi\otimes\dots\otimes\nabla_{i_T}\phi) \tag{36}$$

Observe that, modulo complete contractions of length $\geq\frac{n}{2}+1$, $C^r_{g^n}(\phi) = SC^r_{g^n}(\phi)$.

For any $l \in L^{0,\frac{n}{2},T}$, we write out $Tail[C^l_{g^n}(\phi)]$ as a linear combination of complete contractions in the form (25). We have that:

$$Tail[C^l_{g^n}(\phi)] = \Sigma_{r\in R^l} a_r SC^r_{g^n}(\phi) + \Sigma_{j\in J} a_j C^j_{g^n}(\phi) \tag{37}$$

modulo complete contractions of length $\geq\frac{n}{2}+1$. Each complete contraction $C^j_{g^n}(\phi)$ has length $\frac{n}{2}$ and less than T factors $\nabla\phi$.

Now, for any complete contraction $C^l_{g^n}(\phi)$, $l\in L^{0,\frac{n}{2},\lambda}$ where $\lambda < T$, we have that:

$$Tail[C^l_{g^n}(\phi)] = \Sigma_{v\in V} a_v C^v_{g^n}(\phi)$$

where each complete contraction $C^v_{g^n}(\phi)$ has either length $\geq\frac{n}{2}+1$ or has length $\frac{n}{2}$ but less than T factors $\nabla\phi$. This follows from formula (27).

The super divergence formula can be expressed as:

$$\begin{aligned}&\Sigma_{\lambda=0}^{T-1}\Sigma_{l\in L^{0,\frac{n}{2},\lambda}} a_l Tail[C^l_{g^n}(\phi)] + \Sigma_{l\in L^{0,\frac{n}{2},T}} a_l Tail[C^l_{g^n}(\phi)]+\\&\Sigma_{\lambda=T+1}^{\frac{n}{2}-1}\Sigma_{l\in L^{0,\frac{n}{2},\lambda}} a_l Tail[C^l_{g^n}(\phi)] = 0\end{aligned} \tag{38}$$

modulo complete contractions of length $\geq \frac{n}{2}+1$.

We consider, in (38), the sublinear combination $supdiv[I_{g^n}]|_{\nabla\phi=T}$ of complete contractions of length $\frac{n}{2}$ with T factors $\nabla\phi$. From Lemma 3, we have that

$$supdiv[I_{g^n}]|_{\nabla\phi=T}=0 \tag{39}$$

Furthermore, in view of formula (38) and our observations above, we have the following: Let $\Sigma_{\lambda=T+1}^{\frac{n}{2}-1}\Sigma_{l\in L^{0,\frac{n}{2},\lambda}}a_l Tail[C^l_{g^n}(\phi)]|_{\nabla\phi=T}$ denote the sublinear combination in $\Sigma_{\lambda=T+1}^{\frac{n}{2}-1}\Sigma_{l\in L^{0,\frac{n}{2},\lambda}}a_l Tail[C^l_{g^n}(\phi)]$ of complete contractions with T factors $\nabla\phi$, then:

$$sdI_{\nabla\phi=T}=\Sigma_{\lambda=T+1}^{\frac{n}{2}-1}\Sigma_{l\in L^{0,\frac{n}{2},\lambda}}a_l Tail[C^l_{g^n}(\phi)]|_{\nabla\phi=T}+\Sigma_{l\in L^{0,\frac{n}{2},T}}a_l[\Sigma_{r\in R^l}a_r SC^r_{g^n}(\phi)]=0 \tag{40}$$

Now, by our inductive hypothesis, we are assuming that we know the sublinear combination $\Sigma_{\lambda=T+1}^{\frac{n}{2}-1}\Sigma_{l\in L^{0,\frac{n}{2},\lambda}}a_l C^l_{g^n}(\phi)$. Hence, we deduce that we can determine the sublinear combination $\Sigma_{\lambda=T+1}^{\frac{n}{2}-1}\Sigma_{l\in L^{0,\frac{n}{2},\lambda}}a_l Tail[C^l_{g^n}(\phi)]$. Therefore, we can also determine the sublinear combination $\Sigma_{\lambda=T+1}^{\frac{n}{2}-1}\Sigma_{l\in L^{0,\frac{n}{2},\lambda}}a_l Tail[C^l_{g^n}(\phi)]|_{\nabla\phi=T}$, and using (40), we determine the sublinear combination $\Sigma_{l\in L^{0,\frac{n}{2},T}}a_l[\Sigma_{r\in R^l}a_r SC^r_{g^n}(\phi)]$.

A notational convention: When we write $(\nabla)^a$ we will mean that we are taking *one* covariant derivative ∇_a and then raising the index a. (This is to distinguish from ∇^a which stands for a iterated covariant derivatives). We will now give the following values to factors of the complete contractions in (40): To each factor $\nabla^2_{ab}\phi$ we give the value of $-P_{ab}(x_0)$. Also, to each expression of the from $S\nabla^p_{r_1\dots r_p}\phi(\nabla)^{r_{i_1}}\phi\dots(\nabla)^{r_{i_{p-2}}}\phi$ (where $\{c,d\}=\{r_1,\dots,r_p\}\setminus\{r_{i_1},\dots,r_{i_{p-2}}\}$) we give the value $-P_{cd}\cdot(P^a_a)^{p-2}$. For that assignment A of values, we have that:

$$(n-T)^T\cdot\Sigma_{l\in L^{0,\frac{n}{2},T}}a_l C^l(g^n)+A\{\Sigma_{\lambda=T+1}^{\frac{n}{2}-1}\Sigma_{l\in L^{0,\frac{n}{2},\lambda}}a_l Tail[C^l_{g^n}(\phi)]|_{\nabla\phi=T}\}=0$$

This concludes the proof of Lemma 1. □

4 The harder step: Proof of Lemma 2.

We want to determine the coefficients of the various complete contractions $C^l(g^n)$, indexed in $L^{A_1,\frac{n}{2}-A_1}$.

We consider $I^{\frac{n}{2}-A_1}_{g^n}(\phi)$. For any $C^l(g^n)$, $l\in L^{A_1,\frac{n}{2}-A_1}$, we define $C^l_{g^n}(\phi)$ to be the complete contraction which is obtained from $C^l(g^n)$ by substituting each factor P_{ab} by $-\nabla_{ab}\phi$. We then have that:

$$I^{\frac{n}{2}-A_1}_{g^n}(\phi) = \Sigma_{l\in L^{A_1,\frac{n}{2}-A_1}} a_l C^l_{g^n}(\phi) + \Sigma_{g\in G} a_g C^g_{g^n}(\phi)$$

modulo complete contractions of length $\geq \frac{n}{2}+1$. The complete contractions $C^g_{g^n}(\phi)$ are in the form (25) and they arise from the sublinear combination $\Sigma^{\frac{n}{2}}_{k=\frac{n}{2}-A_1+1}\Sigma_{l\in L^{\frac{n}{2}-k,k}} a_l C^l(g^n)$. Hence, we have that the sublinear combination $\Sigma_{g\in G} a_g C^g_{g^n}(\phi)$ is known.

The complete contractions $C^l_{g^n}(\phi)$ are in the form :

$$contr(W_{i_1j_1k_1l_1}\otimes\cdots\otimes W_{i_{A_1}j_{A_1}k_{A_1}l_{A_1}}\otimes\nabla^2_{a_1b_1}\phi\otimes\cdots\otimes\nabla^2_{a_{\frac{n}{2}-A_1}b_{\frac{n}{2}-A_1}}\phi) \quad (41)$$

While we write the complete contractions $C^g_{g^n}(\phi)$ in the form:

$$contr(R_{i_1j_1k_1l_1}\otimes\cdots\otimes R_{i_{A_1}j_{A_1}k_{A_1}l_{A_1}}\otimes\nabla^2_{a_1b_1}\phi\otimes\cdots\otimes\nabla^2_{a_{\frac{n}{2}-A_1}b_{\frac{n}{2}-A_1}}\phi) \quad (42)$$

(for this equation, the factors R_{ijkl}, $\nabla^2_{ab}\phi$ are allowed to have internal contractions).

Now, we write $\Sigma_{l\in L^{A_1,\frac{n}{2}-A_1}} a_l C^l_{g^n}(\phi)$ as a linear combination:

$$\Sigma_{l\in L^{A_1,\frac{n}{2}-A_1}} a_l C^l_{g^n}(\phi) = \Sigma_{u\in U} a_u C^u_{g^n}(\phi) \quad (43)$$

where each $C^u_{g^n}(\phi)$ is in the form:

$$\begin{aligned}&contr(R_{ijkl}\otimes\dots R_{i'j'k'l'}\otimes Ric_{kl}\otimes\cdots\otimes Ric_{k'l'}\otimes R\otimes\cdots\otimes R\otimes\\&\nabla^2_{\alpha\beta}\phi\otimes\cdots\otimes\nabla^2_{\alpha'\beta'}\phi\otimes\Delta\phi\otimes\cdots\otimes\Delta\phi)\end{aligned} \quad (44)$$

When we employ the above notation we will imply that each of the factors R_{ijkl}, Ric_{ab} and $\nabla^2_{\alpha\beta}\phi$ does not have any of the indices i,j,k,l or a,b or α,β contracting between themselves. Let Z stand for the number of factors R_{ijkl}, X for the number of factors Ric_{ab}, C for the number of factors R, Γ for the number of factors $\nabla^2_{\alpha\beta}\phi$ and Δ for the number of factors $\Delta\phi$. We have that $Z+X+C=A_1$ and $\Gamma+\Delta=\frac{n}{2}-A_1$.

We denote the corresponding index set in U by $U^{Z,X,C,\Gamma,\Delta}$. We then claim the following:

Lemma 5 *Under the assumptions of Lemma 2, we claim that we can determine all the sublinear combinations $\Sigma_{u\in U^{Z,X,C,\Gamma,\Delta}} a_u C^u_{g^n}(\phi)$ above.*

Before we prove this Lemma, let us explain how we can deduce our desired Lemma 2 from Lemma 5.

If we can determine all the sublinear combinations $\Sigma_{u\in U^{Z,X,C,\Gamma,\Delta}} a_u C^u_{g^n}(\phi)$, we then will have determined the whole linear combination $\Sigma_{u\in U} a_u C^u_{g^n}(\phi)$, and hence by (43) we will have determined the linear combination $\Sigma_{l\in L^{A_1,\frac{n}{2}-A_1}} a_l C^l_{g^n}(\phi)$.

But then, setting $\nabla^2_{ab}\phi(x_0) = -P_{ab}(x_0)$, we determine $\Sigma_{l\in L^{A_1,\frac{n}{2}-A_1}} a_l C^l(g^n)$, and hence we will have shown our Lemma.

4.1 The long induction: The Proof of Lemma 5.

We will determine the various sublinear combinations by an induction.

We initially determine the sublinear combination $\Sigma_{u\in U^{0,1,A_1-1,1,\frac{n}{2}-A_1-1}}a_uC^u_{g^n}(\phi)$. By definition, we see that the sublinear combination in question will be of the form $(const)\cdot C^*_{g^n}(\phi)$, where $C^*_{g^n}(\phi)$ is the complete contraction:

$$contr(R^{A_1-1}\otimes Ric^{ab}\otimes\nabla^2_{ab}\phi\otimes(\Delta\phi)^{\frac{n}{2}-A_1-1}) \tag{45}$$

(Thus, determining $\Sigma_{u\in U^{0,1,A_1-1,1,\frac{n}{2}-A_1-1}}a_uC^u_{g^n}(\phi)$ amounts to determining $(const)$).

Then, we will determine the sublinear combination $\Sigma_{u\in U^{0,0,A_1,0,\frac{n}{2}-A_1}}a_uC^u_{g^n}(\phi)$. We observe that this sublinear combination will be in the form:

$$(const)'\cdot contr(R^{A_1}\otimes(\Delta\phi)^{\frac{n}{2}-A_1}) \tag{46}$$

(Thus again, we only have to determine $(const)'$).

Finally, having determined the two sublinear combinations above, we will prove the following inductive statement: Let us suppose that for some number Δ_1+1, we have determined all the sublinear combinations $\Sigma_{u\in U^{Z,X,C,\Gamma,\Delta}}a_uC^u_{g^n}(\phi)$ with $\Delta\geq\Delta_1+1$. Moreover, we assume that for some number C_1+1, we have determined all the sublinear combinations $\Sigma_{u\in U^{Z,X,C,\frac{n}{2}-A_1-\Delta_1,\Delta_1}}a_uC^u_{g^n}(\phi)$ with $C\geq C_1+1$. Finally, we suppose that for some number X_1+1, we have determined all the sublinear combinations $\Sigma_{u\in U^{Z,X,C_1,\frac{n}{2}-A_1-\Delta_1,\Delta_1}}a_uC^u_{g^n}(\phi)$ with $X\geq X_1+1$. We then claim that we can determine the sublinear combination $\Sigma_{u\in U^{A_1-X_1-C_1,X_1,C_1,\frac{n}{2}-A_1-\Delta_1,\Delta_1}}a_uC^u_{g^n}(\phi)$. If we can show the above then by induction we will have proven our Lemma 5.

Before proceeding with the proof, we make note of how the Weyl tensor can be decomposed:

$$\begin{aligned}W_{ijkl}=R_{ijkl}+\frac{1}{n-2}[Ric_{ik}g^n_{jl}+Ric_{jl}g^n_{ik}-Ric_{il}g^n_{jk}-Ric_{jk}g^n_{il}]\\-\frac{R}{(n-1)(n-2)}g^n_{ik}g^n_{jl}+\frac{R}{(n-1)(n-2)}g^n_{il}g^n_{jk}\end{aligned} \tag{47}$$

Determining the sublinear combination $\Sigma_{u\in U^{0,1,A_1-1,1,\frac{n}{2}-A_1-1}}a_uC^u_{g^n}(\phi)$*:*

We consider $I^{\frac{n}{2}-A_1+1}_{g^n}(\phi)$. We focus on the sublinear combinations of complete contractions of length $\frac{n}{2}$ or $\frac{n}{2}+1$ in $I^{\frac{n}{2}-A_1+1}_{g^n}(\phi)$, which we respectively denote by $I^{\frac{n}{2}-A_1+1}_{g^n}(\phi)|_{\frac{n}{2}}$, $I^{\frac{n}{2}-A_1+1}_{g^n}(\phi)|_{\frac{n}{2}+1}$. Using the transformation law (19) and the conformal invariance of the Weyl tensor, we deduce that the sublinear combination $I^{\frac{n}{2}-A_1+1}_{g^n}(\phi)|_{\frac{n}{2}}$ arises from the sublinear combination $\Sigma^{A_1-1}_{B=0}\Sigma_{l\in L^{B,\frac{n}{2}-B}}a_lC^l(g^n)$ in $P(g^n)$. Therefore by our inductive hypothesis, we have that the sublinear combination $I^{\frac{n}{2}-A_1+1}_{g^n}(\phi)|_{\frac{n}{2}}$ in $I^{\frac{n}{2}-A_1+1}_{g^n}(\phi)$ is known.

Now, we also claim that the sublinear combination $I^{\frac{n}{2}-A_1+1}_{g^n}(\phi)|_{\frac{n}{2}+1}$ in $I^{\frac{n}{2}-A_1+1}_{g^n}(\phi)$ can be written as:

$$I^{\frac{n}{2}-A_1+1}_{g^n}(\phi)|_{\frac{n}{2}+1} = \Sigma_{k\in K} a_k C^k_{g^n}(\phi) + \Sigma_{u\in U_1} a_u C^u_{g^n}(\phi) \tag{48}$$

where $\Sigma_{k\in K} a_k C^k_{g^n}(\phi)$ arises from $\Sigma_{B=0}^{A_1-1}\Sigma_{l\in L^{B,\frac{n}{2}-B}} a_l C^l(g^n)$ in $P(g^n)$ and $\Sigma_{u\in U_1} a_u C^u_{g^n}(\phi)$ arises from the sublinear combination $\Sigma_{l\in L^{A_1,\frac{n}{2}-A_1}} a_l C^l(g^n)$ in $P(g^n)$. This means that the contractions $C^l(g^n)$, $l \in L^{B,\frac{n}{2}-B}$ with $B \geq A_1+1$ will not contribute to $I^{\frac{n}{2}-A_1+1}_{g^n}(\phi)|_{\frac{n}{2}+1}$. This follows by virtue of (19). Hence, we may assume that the sublinear combination $\Sigma_{k\in K} a_k C^k_{g^n}(\phi)$ is known.

Now, we initially have that the complete contractions $C^u_{g^n}(\phi)$ on the right hand side of the above are in the form:

$$contr(W_{i_1j_1k_1l_1}\otimes\dots\otimes W_{i_{A_1}j_{A_1}k_{A_1}l_{A_1}}\otimes\nabla^2_{a_1b_1}\phi\otimes\dots\otimes\nabla^2_{a_{\frac{n}{2}-A_1-1}b_{\frac{n}{2}-A_1-1}}\phi\otimes\nabla_x\phi\otimes\nabla_d\phi) \tag{49}$$

Then, we decompose the Weyl tensor as in (47) and we write the linear combination on the right hand side of the above as a linear combination of complete contractions in the form:

$$contr(R_{i_1j_1k_1l_1} \otimes\dots\otimes R_{i_zj_zk_zl_z} \otimes Ric_{h_1e_1} \otimes\dots\otimes Ric_{h_ye_y} \otimes R^q \otimes \nabla^2_{a_1b_1}\phi \otimes \dots \otimes \nabla^2_{a_qb_q}\phi \otimes \nabla_x\phi \otimes \nabla_d\phi \otimes (\Delta\phi)^r) \tag{50}$$

where we are making the notational convention that no two indices in any factor $R_{ijkl}, Ric_{ab}, \nabla^2_{ij}\phi$ are contracting between themselves. We write:

$$\Sigma_{u\in U_1} a_u C^u_{g^n}(\phi) = \Sigma_{u\in U_2} a_u C^u_{g^n}(\phi)$$

where each $C^u_{g^n}(\phi)$, $u \in U_2$ is in the form (50). We replace the expression $\Sigma_{u\in U_2} a_u C^u_{g^n}(\phi)$ for $\Sigma_{u\in U_1} a_u C^u_{g^n}(\phi)$ in (48). Moreover, we assume that each $C^k_{g^n}(\phi)$ in (48) is in the form (50).

Now, we focus on the sublinear combination in $\Sigma_{u\in U_2} a_u C^u_{g^n}(\phi)$ that consists of complete contractions in the form (50) with $Z = 0$ factors R_{ijkl}, $Y = 1$ factor Ric_{he}, $C = A_1 - 1$ factors R, $\Gamma = 0$ factors $\nabla^2\phi$, $\Delta = \frac{n}{2} - A_1 - 1$ factors $\Delta\phi$. We also assume that the two factors $\nabla\phi$ contract against the two indices of the one factor Ric_{ij}. Therefore, we have that the sublinear combination in question is of the form $(const)_* \cdot C^*_{g^n}(\phi)$, where $C^*_{g^n}(\phi)$ is in the form:

$$contr(R^{A_1-1} \otimes Ric^{ij} \otimes \nabla_i\phi \otimes \nabla_j\phi \otimes (\Delta\phi)^{\frac{n}{2}-A_1-1}) \tag{51}$$

We now make two claims:

Lemma 6 *We have that the sublinear combination* $(const)_* \cdot C^*_{g^n}(\phi)$ *in* $\Sigma_{u\in U_2} a_u C^u_{g^n}(\phi)$ *arises from the sublinear combination* $\Sigma_{u\in U^{0,1,A_1-1,1,\frac{n}{2}-A_1-1}} a_u C^u_{g^n}(\phi)$ *in* $I^{\frac{n}{2}-A_1}_{g^n}(\phi)$ *by replacing the factor* $\nabla^2_{ij}\phi$ *by an expression* $-\nabla_i\phi\nabla_j\phi$.

Our second claim is that the sublinear combination $(const)_* \cdot C^*_{g^n}(\phi)$ *can be determined from the known sublinear combinations in (48), using the shadow divergence formula for* $I^{\frac{n}{2}-A_1+1}_{g^n}(\phi)$.

We observe that if we can show the above Lemma, we will then have determined the sublinear combination $\Sigma_{u\in U^{0,1,A_1-1,1,\frac{n}{2}-A_1-1}} a_u C^u_{g^n}(\phi)$ in $I^{\frac{n}{2}-A_1}_{g^n}(\phi)$, and hence proven the first base case of our induction.

Proof of Lemma 6: We begin with the first part. Initially, let us focus on the sublinear combination $\Sigma_{u\in U^{0,1,A_1-1,1,\frac{n}{2}-A_1-1}} a_u C^u_{g^n}(\phi)$ in $I^{\frac{n}{2}-A_1}_{g^n}(\phi)$ and understand in detail how it arises. For each $l \in L^{A_1,\frac{n}{2}-A_1}$, we consider the complete contraction $C^l_{g^n}(\phi)$ defined above, which will be in the form (41). We then decompose the factors W_{ijkl} as in (47).

Now, for each factor W_{ijkl}, we have the option of replacing it by one of the 7 expressions on the right hand side of (47). Therefore, we can write $C^l_{g^n}(\phi)$ as a sum of 7^{A_1} complete contractions in the form (42):

$$C^l_{g^n}(\phi) = \Sigma^{7^{A_1}}_{\tau=1} a_\tau C^\tau_{g^n}(\phi) \tag{52}$$

Each of the 7^{A_1} different summands corresponds to a different sequence of substitutions of the A_1 factors W_{ijkl} as explained above. We then group up the complete contractions $C^\tau_{g^n}(\phi)$ on the right hand side of the above that are of the form (45), and we denote that sublinear combination in (52) by $F[C^l_{g^n}(\phi)]$. Hence, using this notation we have that:

$$\Sigma_{u\in U^{0,1,A_1-1,1,\frac{n}{2}-A_1-1}} a_u C^u_{g^n}(\phi) = \Sigma_{l\in L^{A_1,\frac{n}{2}-A_1}} a_l F[C^l_{g^n}(\phi)]$$

Now, we consider the complete contractions in $Image^{\frac{n}{2}-A_1+1}_\phi[C^l(g^n)]$, for each $l \in L^{A_1,\frac{n}{2}-A_1}$. We are only interested in the sublinear combination

$$Image^{\frac{n}{2}-A_1+1}_\phi|_{\frac{n}{2}+1}[C^l(g^n)]$$

of complete contractions of length $\frac{n}{2}+1$. It follows that this sublinear combination arises by replacing $\frac{n}{2}-A_1-1$ factors P_{ab} by the expression $-\nabla^2_{ab}\phi$ on the right hand side of (19) and also by replacing one factor P_{ab} by a quadratic expression on the right hand side of (19).

Now, we further denote by $Image^{\frac{n}{2}-A_1+1,+}_\phi|_{\frac{n}{2}+1}[C^l(g^n)]$ the sublinear combination in $Image^{\frac{n}{2}-A_1+1}_\phi|_{\frac{n}{2}+1}[C^l(g^n)]$ that arises when we replace $\frac{n}{2}-A_1-1$ factors P_{ab} by $-\nabla^2_{ab}\phi$ and one factor P_{ab} by the expression $g_{ab}|\nabla\phi|^2$. We trivially observe that if we write out $Image^{\frac{n}{2}-A_1+1,\sigma+1,+}_\phi[C^l(g^n)]$ as a linear combination of complete contractions in the form (50), none will be in the form (51).

Hence, we may restrict our attention to the sublinear combination $Image_{\phi}^{\frac{n}{2}-A_1+1,-}[C^l(g^n)]$ in $Image_{\phi}^{\frac{n}{2}-A_1+1}[C^l(g^n)]$ that arises when we replace $\frac{n}{2}-A_1-1$ factors P_{ab} by $-\nabla^2_{ab}\phi$ and one factor P_{ab} by $\nabla_a\phi\nabla_b\phi$. Hence, comparing $Image^{\frac{n}{2}-A_1}\{\Sigma_{l\in L^{A_1,\frac{n}{2}-A_1}}a_lC^l(g^n)\}$ and $Image^{\frac{n}{2}-A_1+1,-}\{\Sigma_{l\in L^{A_1,\frac{n}{2}-A_1}}a_lC^l(g^n)\}$, we see that $Image^{\frac{n}{2}-A_1+1,-}\{\Sigma_{l\in L^{A_1,\frac{n}{2}-A_1}}a_lC^l(g^n)\}$ arises from $Image^{\frac{n}{2}-A_1}\{\Sigma_{l\in L^{A_1,\frac{n}{2}-A_1}}a_lC^l(g^n)\}$ by picking out one factor $\nabla^2_{ab}\phi$ from each complete contraction in the form (41) in $Image^{\frac{n}{2}-A_1}\{\Sigma_{l\in L^{A_1,\frac{n}{2}-A_1}}a_lC^l(g^n)\}$ (this factor may now also be of the form $\Delta\phi$) and replacing it by an expression $-\nabla_a\phi\nabla_b\phi$. In that case, if we repeat the decomposition of the factors W_{ijkl} to the complete contractions in $Image_{\phi}^{\frac{n}{2}-A_1+1,-}[C^l(g^n)]$, we obtain the first claim of our Lemma.

Now, for the second part of our Lemma, we first of all denote $(const)_*C^*_{g^n}(\phi)$ by $\Sigma_{u\in U_2^*}a_uC^u_{g^n}(\phi)$. We then want to apply the shadow divergence formula to $I^{\frac{n}{2}-A_1+1}_{g^n}(\phi)$ and determine the sublinear combination $\Sigma_{u\in U_2^*}a_uC^u_{g^n}(\phi)$. We will focus on the sublinear combination of $\vec{\xi}$-contractions in $Shad[I^{\frac{n}{2}-A_1+1}_{g^n}(\phi)]$ that are in the form:

$$contr((|\vec{\xi}|^2)^{A_1-1}\otimes S\nabla^{\frac{n}{2}-A_1}_{r_1\dots r_{\frac{n}{2}-A_1}}\vec{\xi}_{r_{\frac{n}{2}-A_1+1}}\otimes\nabla^{r_1}\phi\otimes\dots\otimes\nabla^{r_{\frac{n}{2}-A_1+1}}\phi) \quad (53)$$

If we denote the sublinear combination of those $\vec{\xi}$-contractions in $Shad[I^{\frac{n}{2}-A_1+1}_{g^n}(\phi)]$ by $Shad_+[I^{\frac{n}{2}-A_1+1}_{g^n}(\phi)]$, we claim that:

$$Shad_+[I^{\frac{n}{2}-A_1+1}_{g^n}(\phi)]=0 \quad (54)$$

This is straightforward because the shadow divergence formula holds formally. Now, for each $k\in K$ (see (48)) we denote by $Tail^{Shad}_+[C^k_{g^n}(\phi)]$ the sublinear combination in each $Tail^{Shad}[C^k_{g^n}(\phi)]$ that consists of $\vec{\xi}$-contractions in the form (53). Analogously, for each $u\in U_2$, we denote by $Tail^{Shad}_+[C^u_{g^n}(\phi)]$ the sublinear combination in each $Tail^{Shad}[C^u_{g^n}(\phi)]$ that consists of $\vec{\xi}$-contractions in the form (54). Now, we observe that the $\vec{\xi}$-length of the $\vec{\xi}$-contraction in (54) is $\frac{n}{2}+1$. Hence, in view of the Lemma on acceptable descendants in [2] and also (48), (54), we deduce that:

$$Shad_+[I^{\frac{n}{2}-A_1+1}_{g^n}(\phi)|_{\frac{n}{2}}]=Tail^{Shad}_+[\Sigma_{k\in K}a_kC^k_{g^n}(\phi)]+Tail^{Shad}_+[\Sigma_{u\in U_2}a_uC^u_{g^n}(\phi)]=0 \quad (55)$$

Therefore, if we could show that for each $u\in U_2\setminus U_2^*$, we have that:

$$Shad_+[C^u_{g^n}(\phi)]=0 \quad (56)$$

we could then use equation (55) to determine the sublinear combination

$$\Sigma_{u\in U_2^*}a_uTail^{Shad}_+[C^u_{g^n}(\phi)].$$

Let us observe how it would then be straightforward to determine $\Sigma_{u\in U_2^*}a_u C^u_{g^n}(\phi)$:

We claim that for $u\in U_2^*$, $Tail^{Shad}_+[C^u_{g^n}(\phi)]=(-1)^{\frac{n}{2}-A_1}C^*_{g^n}(\phi,\vec{\xi})$. To see this, we note that $Tail^{Shad}_+[C^u_{g^n}(\phi)]$ arises in the following way: Let us denote by $C^*_{g^n}(\psi,\vec{\xi})$ the descendant of $C^u_{g^n}(\phi)$ that arises by replacing the A_1-1 factors R by $|\vec{\xi}|^2$, the $\frac{n}{2}-A_1-1$ factors $\Delta\phi$ by $\nabla_i\phi\vec{\xi^i}$ and the factor Ric_{ij} by $-\nabla_i\vec{\xi_j}$ (all in N-cancelled notation). Recall from [2] that $O^{Shad}[C_{g^n}(\phi,\vec{\xi})]$ stands for the sublinear combination of hard and stigmatized $\vec{\xi}$-contractions (of both types) that arise along the iterative integrations by parts of the $\vec{\xi}$-contraction $C_{g^n}(\phi,\vec{\xi})$. We now define $O^{Shad}_+[C_{g^n}(\phi,\vec{\xi})]$ to stand for the sublinear combination of those $\vec{\xi}$-contractions that are in the form (53).

We then claim (claim 1) that $O^{Shad}_+[C^*_{g^n}(\phi,\vec{\xi})]=(-1)^{\frac{n}{2}-A_1}C^*_{g^n}(\phi,\vec{\xi})$ (the left hand side here stands for the sublinear combination of complete contractions in the form $C^*_{g^n}(\phi,\vec{\xi})$ in $O^{Shad}[C^*_{g^n}(\phi,\vec{\xi})]$). Moreover, we claim (claim 2) that for any other descendant $C^d_{g^n}(\phi,\vec{\xi})$ of $C^*_{g^n}(\phi)$ we will have $O^{Shad}_+[C^d_{g^n}(\phi,\vec{\xi})]=0$.

The second claim follows by simply observing that $C^d_{g^n}(\phi,\vec{\xi})$ must contain a factor with an internal contraction, hence each $\vec{\xi}$-contraction in $O^{Shad}[C^d_{g^n}(\phi,\vec{\xi})]$ with length $\frac{n}{2}+1$ must have a factor with an internal contraction. Our first claim follows by integrating by parts all the factors $\vec{\xi_i}$ that contract against factors $\nabla\phi$ and making all the derivatives ∇_i hit the factor $\nabla\vec{\xi}$ and then symmetrizing. We observe that any other $\vec{\xi}$-contraction that arises in the iterative integration by parts will not be of the form $C^*_{g^n}(\phi,\vec{\xi})$: It will either have $\vec{\xi}$-length $\geq\frac{n}{2}+2$ or a factor $\nabla^a\phi, a\geq 2$ or less than A_1-1 factors $|\vec{\xi}|^2$.

In view of the above, and since (55) holds formally, if we replace each expression $S\nabla^{\frac{n}{2}-A_1}_{r_1\dots r_{\frac{n}{2}-A_1}}\vec{\xi}_{r_{\frac{n}{2}-A_1+1}}\otimes\nabla^{r_1}\phi\otimes\dots\otimes\nabla^{r_{\frac{n}{2}-A_1+1}}\phi$ in each complete contraction in (55) by $\nabla^i\phi\nabla^j\phi\nabla_i\vec{\xi_j}(\Delta\phi)^{\frac{n}{2}-A_1-1}$ and each factor $|\vec{\xi}|^2$ by a factor R, we can then determine $\Sigma_{u\in U_2^*}a_u C^u_{g^n}(\phi)$. Hence, showing (56) would complete the proof of our Lemma.

But (56) is easy to prove: Let us suppose that $C^u_{g^n}(\phi)$ is in the form (50) and has less than A_1-1 factors R. It then follows that each descendent $C^{u,l}_{g^n}(\phi,\vec{\xi})$ of $C^u_{g^n}(\phi)$ will have less than A_1-1 factors $|\vec{\xi}|^2$ (by the Lemma on the acceptable descendants in [2]) and hence, by the iterative integration by parts procedure, each $\vec{\xi}$-contraction in $Tail^{Shad}[C^{u,l}_{g^n}(\phi,\vec{\xi})]$ will have less than A_1-1 factors $|\vec{\xi}|^2$ (by Lemma 15 in [2]) and hence we have shown (56) in this case. Now, we consider the case where $C^u_{g^n}(\phi)$ is in the form (50) and has less than $\frac{n}{2}-A_1-1$ factors $\Delta\phi$, and hence has at least one factor $\nabla^2\phi\neq\Delta\phi$. It then follows that each descendent $C^{u,l}_{g^n}(\phi,\vec{\xi})$ of $C^u_{g^n}(\phi)$ will have at least one factor $\nabla^2\phi\neq\Delta\phi$ (by the Lemma on the acceptable descendants in [2]). Hence, we have that each $\vec{\xi}$-contraction of $\vec{\xi}$-length $\frac{n}{2}+1$ in $Tail^{Shad}[C^u_{g^n}(\phi)]$ will have at least one factor $\nabla^a\phi, a\geq 2$ and therefore $Tail^{Shad}_+[C^u_{g^n}(\phi)]=0$.

We are thus left with the case where $u\in U_2\setminus U_2^*$ and $C^u_{g^n}(\phi)$ has at least A_1-1 factors R and at least $\frac{n}{2}-A_1-1$ factors $\Delta\phi$. It then follows that $C^u_{g^n}(\phi)$

must be in the form:

$$contr(R^{A_1}\otimes(\Delta\phi)^{\frac{n}{2}-A_1-1}\otimes|\nabla\phi|^2)$$

But then, by the iterative integrations by parts procedure, we observe that each $\vec{\xi}$-contraction of $\vec{\xi}$-length $\frac{n}{2}+1$ in $Tail^{Shad}[C^u_{g^n}(\phi)]$ will either have a factor $\nabla^a\phi, a\ge 2$ or will have two factors $\nabla\phi$ that contract against each other. Therefore, we again have our desired (56) in this case. We have shown our Lemma. $\Box$

Determining the sublinear combination $\Sigma_{u\in U^{0,0,A_1,0,\frac{n}{2}-A_1}} a_u C^u_{g^n}(\phi)$*:*

We consider the shadow divergence formula of $I^{\frac{n}{2}-A_1}_{g^n}(\phi)$, $Shad[I^{\frac{n}{2}-A_1}_{g^n}(\phi)]$. We focus on the sublinear combination of $\vec{\xi}$-contractions in the form:

$$contr((|\vec{\xi}|^2)^{A_1-1}\otimes\nabla^{\frac{n}{2}-A_1+1}_{r_1\dots r_{\frac{n}{2}-A_1-1}i}\vec{\xi}_j\otimes\nabla^{ij}\phi\otimes(\nabla)^{r_1}\phi\otimes\dots\otimes(\nabla)^{r_{\frac{n}{2}-A_1-1}}\phi) \quad (57)$$

We denote the above $\vec{\xi}$-contraction by $C^{\sharp}_{g^n}(\phi,\vec{\xi})$ for short. For each $C_{g^n}(\phi)$ in the form (42) of length $\frac{n}{2}$, we denote by $Tail^{Shad}_+[C_{g^n}(\phi)]$ the sublinear combination of $\vec{\xi}$-contractions in the form (57) in $Tail^{Shad}[C_{g^n}(\phi)]$. (Note that we are changing the meaning of $Tail^{Shad}_+[C_{g^n}(\phi)]$). This notation extends to linear combinations. Now, since the Shadow divergence formula holds formally, we will have that:

$$Tail^{Shad}_+[I^{\frac{n}{2}-A_1}_{g^n}(\phi)]=0$$

We write out $I^{\frac{n}{2}-A_1}_{g^n}(\phi)$ in the form:

$$I^{\frac{n}{2}-A_1}_{g^n}(\phi)=\Sigma_{k\in K}a_kC^k_{g^n}(\phi)+\Sigma_{u\in U}a_uC^u_{g^n}(\phi) \quad (58)$$

modulo complete contractions of length $\ge\frac{n}{2}+1$. Here $\Sigma_{k\in K}a_kC^k_{g^n}(\phi)$ arises from the sublinear combination $\Sigma^{A_1-1}_{A=0}\Sigma_{l\in L^{A,\frac{n}{2}-A}}a_lC^l(g^n)$. Hence, we have that $\Sigma_{k\in K}a_kC^k_{g^n}(\phi)$ is known. We note that the index set K differs from K in (48). We deduce that:

$$\Sigma_{k\in K}a_kTail^{Shad}_+[C^k_{g^n}(\phi)]+\Sigma_{u\in U}a_uTail^{Shad}_+[C^u_{g^n}(\phi)]=0 \quad (59)$$

Now, we claim that for each $u\in U\setminus(U^{0,0,A_1,0,\frac{n}{2}-A_1}\bigcup U^{0,1,A_1-1,1,\frac{n}{2}-A_1-1})$ we have that $Tail^{Shad}_+[C^u_{g^n}(\phi)]=0$. This follows by a similar reasoning as for the previous case: For each u above, we have that either $C^u_{g^n}(\phi)$ has less than A_1-1 factors R or it has less than $\frac{n}{2}-A_1-1$ factors $\Delta\phi$. In the first case we then have that each $\vec{\xi}$-contraction in $Tail^{Shad}[C^u_{g^n}(\phi)]$ will have less than A_1-1 factors $|\vec{\xi}|^2$ and in the second, it will have less than $\frac{n}{2}-A_1-1$ factors $\nabla\phi$.

In view of this fact, we can then use (59) to determine the sublinear combination $\Sigma_{u\in U^{0,0,A_1,0,\frac{n}{2}-A_1}}a_uTail^{Shad}_+[C^u_{g^n}(\phi)]$.

We then claim that knowing $\Sigma_{u\in U^{0,0,A_1,0,\frac{n}{2}-A_1}} a_u$ $Tail_+^{Shad}[C^u_{g^n}(\phi)]$ we can determine $\Sigma_{u\in U^{0,0,A_1,0,\frac{n}{2}-A_1}} a_u C^u_{g^n}(\phi)$. Specifically, we show that for the complete contraction $C^+_{g^n}(\phi) = C^u_{g^n}(\phi)$, $u \in U^{0,0,A_1,0,\frac{n}{2}-A_1}$, we have that:

$$Tail_+^{Shad}[C^+_{g^n}(\phi)] = (-1)^{\frac{n}{2}-A_1-1} 2A_1 \cdot (\frac{n}{2} - A_1) \cdot C^\sharp_{g^n}(\phi, \vec{\xi}) \tag{60}$$

Proof of (60):

For any $\vec{\xi}$-contraction $C_{g^n}(\phi, \vec{\xi})$, we denote by $O_+^{Shad}[C_{g^n}(\phi, \vec{\xi})]$ the sublinear combination in $O^{Shad}[C_{g^n}(\phi, \vec{\xi})]$ of $\vec{\xi}$-contractions in the form (57).

Firstly, we denote by $C^+_{g^n}(\phi, \vec{\xi})$ the descendant of $C^+_{g^n}(\phi)$ that arises by replacing each of the A_1 factors R by $|\vec{\xi}|^2$ and each of the $\frac{n}{2} - A_1$ factors $\Delta\phi$ by $\vec{\xi}^i \nabla_i \phi$ (in the N-cancelled notation). We observe that for any descendant $C'_{g^n}(\phi, \vec{\xi})$ of $C^+_{g^n}(\phi)$ other than the above, we will have that $Tail_+^{Shad}[C'_{g^n}(\phi, \vec{\xi})]$=0. This is true by virtue of the same arguments as for the previous case (at $\vec{\xi}$-length $\frac{n}{2}$ there must be an internal contraction). Hence, it suffices to show that $O_+^{Shad}[C^+_{g^n}(\phi, \vec{\xi})]$ is equal to the right hand side of (60).

So, let us begin by performing the iterative integration by parts. We first integrate by parts the factor $\vec{\xi}^i$ that contracts against the first factor $\nabla_i \phi$. Note that, although we have imposed restrictions on the order of our integrations by parts, in this case we can pick an order so that we first integrate by parts with respect to this factor $\vec{\xi}$. If ∇_i hits a factor $\nabla\phi$ or a factor $\vec{\xi}$ that does not contract against another factor $\vec{\xi}$, we denote the $\vec{\xi}$-contraction that is generically thus obtained by $C^d_{g^n}(\phi, \vec{\xi})$. We observe that $O_+^{Shad}[C^d_{g^n}(\phi, \vec{\xi})] = 0$, since each $\vec{\xi}$-contraction in that sublinear combination will either have length $\geq \frac{n}{2} + 1$ or at least one factor $\nabla^a \phi$, $a \geq 2$. If ∇_i hits a factor $|\vec{\xi}|^2$, we obtain an expression $2\nabla_i \vec{\xi}_j \vec{\xi}^j$ and we denote the $\vec{\xi}$-contraction that we have obtained by $C^*_{g^n}(\phi, \vec{\xi})$. We then proceed to integrate by parts the factor $\vec{\xi}^j$.

Now, if ∇^j hits a factor $\vec{\xi}$ that does not contract against another factor $\vec{\xi}$ or if it hits a factor $|\vec{\xi}|^2$, we generically denote the $\vec{\xi}$-contraction that is thus obtained by $C^d_{g^n}(\phi, \vec{\xi})$ and we observe that $O_+^{Shad}[C^d_{g^n}(\phi, \vec{\xi})] = 0$. This follows because in the first case we will obtain a $\vec{\xi}$-contraction of $\vec{\xi}$-length $\geq \frac{n}{2} + 1$ and in the second we will have less than $A_1 - 1$ factors $|\vec{\xi}|^2$.

Initially, we consider the $\vec{\xi}$-contraction $C^{*;1}_{g^n}(\phi, \vec{\xi})$ that arises when ∇^j hits the first factor $\nabla_i \phi$. In that case, $C^{*;1}_{g^n}(\phi, \vec{\xi})$ is the complete contraction:

$$contr((|\vec{\xi}|^2)^{A_1-1} \otimes \nabla^2_i \vec{\xi}_j \otimes \nabla^{ij}\phi \otimes (\nabla)^{r_1} \phi \vec{\xi}_{r_1} \otimes \dots \otimes (\nabla)^{r_{\frac{n}{2}-A_1-1}} \phi \vec{\xi}_{r_{\frac{n}{2}-A_1-1}}) \tag{61}$$

We show that $O_+^{Shad}[C^1_{g^n}(\phi, \vec{\xi})] = (-1)^{\frac{n}{2}-A_1-1} C^\sharp_{g^n}(\phi, \vec{\xi})$.

This follows by the iterative integrations by parts procedure. The algorithm to obtain $(-1)^{\frac{n}{2}-A_1-1}C^{\sharp}_{g^n}(\phi,\vec{\xi})$ is to successively integrate by parts each of the $\frac{n}{2}-A_1-1$ factors $\vec{\xi}$ that contract against a factor $\nabla\phi$ and make it hit the one factor $S\nabla^p\vec{\xi}$ and then symmetrize. We then obtain $(-1)^{\frac{n}{2}-A_1-1}C^{\sharp}_{g^n}(\phi,\vec{\xi})$. We observe that if at any stage we integrate by parts a factor $\vec{\xi}$ and hit the factor $\nabla^2\phi$ or a factor $|\vec{\xi}|^2$ or a factor $\vec{\xi}$ or a factor $\nabla\phi$, then performing the rest of the iterative integrations by parts we will not obtain a $\vec{\xi}$-contraction in the form $C^{\sharp}_{g^n}(\phi,\vec{\xi})$.

On the other hand, we consider the $\vec{\xi}$-contraction that arises when ∇^j hits the h^{th} factor $\nabla\phi$, $h\geq 2$. We denote the $\vec{\xi}$-contraction that arises thus by $C^{*,h}_{g^n}(\phi,\vec{\xi})$. We then claim that $O^{Shad}_{+}[C^{*,h}_{g^n}(\phi,\vec{\xi})]=(-1)^{\frac{n}{2}-A_1-1}C^{\sharp}_{g^n}(\phi,\vec{\xi})$. It is clear that if we can show the above claim, (60) will follow immediately.

To see this, we initially observe that up to permuting factors $\nabla\phi$, $C^{*,h}_{g^n}(\phi,\vec{\xi})$ is in the form:

$$contr((|\vec{\xi}|^2)^{A_1-1}\otimes\nabla_i\vec{\xi}_j\otimes(\nabla)^i\phi\otimes(\nabla)^j(\nabla)^{r_1}\phi\vec{\xi}_{r_1}\otimes\dots\otimes(\nabla)^{r_{\frac{n}{2}-A_1-1}}\phi\vec{\xi}_{r_{\frac{n}{2}-A_1-1}}) \tag{62}$$

Moreover, it follows that $(-1)^{\frac{n}{2}-A_1-1}C^{\sharp}_{g^n}(\phi,\vec{\xi})$ arises in $O^{Shad}_{+}[C^{*,h}_{g^n}(\phi,\vec{\xi})]$ when we integrate by parts all the factors $\vec{\xi}_a$ and hit the factor $S\nabla\vec{\xi}$ and then replace $\nabla^{\frac{n}{2}-A_1-1}\nabla\vec{\xi}$ by $S\nabla^{\frac{n}{2}-A_1}\vec{\xi}$. We observe that if we perform any other integration by parts, we will not obtain $C^{\sharp}_{g^n}(\phi,\vec{\xi})$: If we hit a factor $|\vec{\xi}|^2$ by a ∇, we will obtain a $\vec{\xi}$-contraction with fewer than A_1-1 factors $|\vec{\xi}|^2$. If we hit a factor $\vec{\xi}$ that does not contract against another factor $\vec{\xi}$, we will have $\vec{\xi}$-length $\geq\frac{n}{2}+1$. If we hit a factor $\nabla\phi$ or the factor $\nabla^2\phi$, we will respectively have two factors $S\nabla^p\phi$ with $p\geq 2$ or one factor $S\nabla^p\phi$ with $p\geq 3$. Finally, if we hit the factor $S\nabla^p\vec{\xi}$ by a derivative ∇_i and anti-symmetrize using the equation:

$$\begin{gathered}\nabla_a S\nabla^m_{r_1\dots r_m}\vec{\xi}_j=S\nabla^m_{ar_1\dots r_m}\vec{\xi}_j+C_{m-1}\cdot S^*\nabla^{m-1}_{r_1\dots r_{m-1}}R_{aijd}\vec{\xi}^d+\\ \Sigma_{u\in U^m}a_u pcontr(\nabla^{m'}R_{abcd}S\nabla^{s_u}\vec{\xi})\end{gathered} \tag{63}$$

from [2] (and the notational conventions there), we obtain a $\vec{\xi}$-contraction with a factor of the form $\nabla^m R_{ijkl}$. Hence, by the iterative integrations by parts procedure, the O^{Shad} of such a factor will consist of $\vec{\xi}$-contractions with a factor $\nabla^m R_{ijkl}$, so we have completely shown our claim.

Hence, we have determined $\Sigma_{u\in U^{0,0,A_1,0,\frac{n}{2}-A_1}}a_u Tail^{Shad}_{+}[C^u_{g^n}(\phi)]$. In other words, we have determined the constant $(Const)_\sharp$ for which:

$$\Sigma_{u\in U^{0,0,A_1,0,\frac{n}{2}-A_1}}a_u Tail^{Shad}_{+}[C^u_{g^n}(\phi)]=(Const_\sharp)\cdot C^{\sharp}_{g^n}(\phi,\vec{\xi}) \tag{64}$$

Now, we only have to replace each expression $|\vec{\xi}|^2$ by an expression R and the expression

$$\nabla^{\frac{n}{2}-A_1+1}_{r_1\dots r_{\frac{n}{2}-A_1-1}i}\vec{\xi}_j\otimes\nabla^{ij}\phi\otimes(\nabla)^{r_1}\phi\otimes\dots\otimes(\nabla)^{r_{\frac{n}{2}-A_1-1}}\phi$$

by an expression $R\cdot(\Delta\phi)^{\frac{n}{2}-A_1}$. Then, using (60) and (64), we determine the constant $Const'$ for which:

$$\Sigma_{u\in U^{0,0,A_1,0,\frac{n}{2}-A_1}}a_u C^u_{g^n}(\phi)=(Const')\cdot R^{A_1}\cdot(\Delta\phi)^{\frac{n}{2}-A_1}$$

In other words, we determine the sublinear combination $\Sigma_{u\in U^{0,0,A_1,0,\frac{n}{2}-A_1}}a_u C^u_{g^n}(\phi)$. That concludes the proof of our second claim.

4.2 Determining the sublinear combination $\Sigma_{u\in U^{A_1-X_1-C_1,X_1,C_1,\frac{n}{2}-\Delta_1,\Delta_1}}a_u C^u_{g^n}(\phi)$.

We call the list $(A_1-X_1-C_1,X_1,C_1,\frac{n}{2}-A_1-\Delta_1,\Delta_1)$ the *critical list.* We denote the index set $U^{A_1-X_1-C_1,X_1,C_1,\frac{n}{2}-A_1-\Delta_1,\Delta_1}$ by U^{crit} for short. Moreover, whenever we refer to a list (Z,X,C,Γ,Δ) for which we have not yet determined $\Sigma_{u\in U^{Z,X,C,\Gamma,\Delta}}a_u C^u_{g^n}(\phi)$, we will say that the list (Z,X,C,Γ,Δ) is *subsequent* to the critical list. We will also say that u or $C^u_{g^n}(\phi)$ is subsequent to the critical list when $u\in U^{Z,X,C,\Gamma,\Delta}$.

On the other hand, for each list (Z,X,C,Γ,Δ) where we have determined $\Sigma_{u\in U^{Z,X,C,\Gamma,\Delta}}a_u C^u_{g^n}(\phi)$, we will say that the list (Z,X,C,Γ,Δ) *preceded* the critical list. Accordingly, in that case, if $u\in U^{Z,X,C,\Gamma,\Delta}$, we will say that u or $C^u_{g^n}(\phi)$ preceded the critical list.

We will distinguish three cases and separately prove our claim in each of those cases. The first case is when $\Delta_1<\frac{n}{2}-A_1$. The second one is when $\Delta_1=\frac{n}{2}-A_1$ and $X_1>0$. The third is when $\Delta_1=\frac{n}{2}-A_1$, $X_1=0$. In the third case we observe that we will have that $X_1+C_1<A_1$ (otherwise we are in the base case that we have already dealt with). In each of the three cases, we will use the equation:

$$I^{\frac{n}{2}-A_1}_{g^n}(\phi)=\Sigma_{k\in K}a_k C^k_{g^n}(\phi)+\Sigma_{u\in U}a_u C^u_{g^n}(\phi)\tag{65}$$

which holds modulo complete contractions of length $\geq\frac{n}{2}+1$. We recall that the sublinear combination $\Sigma_{k\in K}a_k C^k_{g^n}(\phi)$ is known, and each sublinear combination $U^{Z,X,C,\Gamma,\Delta}$, where (Z,X,C,Γ,Δ) precedes U^{crit} is also known.

We proceed to prove our claim in each of the three cases.

The first case. We consider $Shad[I^{\frac{n}{2}-A_1}_{g^n}(\phi)]$ and focus on the sublinear combination of $\vec{\xi}$-contractions in the following form:

$$contr(R_{i_1j_1k_1l_1}\otimes\dots\otimes R_{i_{A_1-X_1-C_1}j_{A_1-X_1-C_1}k_{A_1-X_1-C_1}l_{A_1-X_1-C_1}}\otimes\nabla_{a_1}\vec{\xi}_{b_1}\otimes\dots\otimes \nabla_{a_{X_1}}\vec{\xi}_{b_{X_1}}\otimes(|\vec{\xi}|^2)^{C_1}\otimes S(\nabla^{\Delta_1}_{s_1\dots s_{\Delta_1}}\nabla^2_{f_1g_1})\phi\otimes\nabla^2_{f_2g_2}\phi\otimes\dots\otimes\nabla^2_{f_{\frac{n}{2}-A_1-\Delta_1}g_{\frac{n}{2}-A_1-\Delta_1}}\phi \otimes(\nabla)^{s_1}\phi\otimes\dots\otimes(\nabla)^{s_{\Delta_1}}\phi) \tag{66}$$

We denote the sublinear combination of $\vec{\xi}$-contractions in the form (66) in $Shad[I^{\frac{n}{2}-A_1}_{g^n}(\phi)]$ by $Shad_o[I^{\frac{n}{2}-A_1}_{g^n}(\phi)]$. We then claim that:

$$Shad_o[I^{\frac{n}{2}-A_1}_{g^n}(\phi)]=0 \tag{67}$$

This can be seen by the following reasoning: We write out the sublinear combination of $\vec{\xi}$-contractions of $\vec{\xi}$-length $\frac{n}{2}$ in $Shad[I^{\frac{n}{2}-A_1}_{g^n}(\phi)]$ as a linear combination of $\vec{\xi}$-contractions in the form:

$$contr(\nabla^{m_1}_{r_1\dots r_{m_1}}R_{i_1j_1k_1l_1}\otimes\dots\otimes\nabla^{m_s}_{v_1\dots v_{m_s}}R_{i_sj_sk_sl_s}\otimes \nabla^{p_1}_{t_1\dots t_{p_1}}Ric_{\alpha_1\beta_1}\otimes\dots\otimes\nabla^{p_q}_{z_1\dots z_{p_q}}Ric_{\alpha_q\beta_q}\otimes\nabla^{\nu_1}_{\chi_1\dots\chi_{\nu_1}}\phi\otimes\dots\otimes\nabla^{\nu_Z}_{\omega_1\dots\omega_{\nu_Z}}\phi \otimes S\nabla^{\mu_1}\vec{\xi}_{j_1}\dots\dots S\nabla^{\mu_r}\vec{\xi}_{j_s}\otimes|\vec{\xi}|^2\otimes\dots\otimes|\vec{\xi}|^2) \tag{68}$$

with $Z=\frac{n}{2}-A_1$. Then, we define $Tail^{Shad}_\alpha[I^{\frac{n}{2}-A_1}_{g^n}(\phi)]$ to stand for the sublinear combination in $Tail^{Shad}[I^{\frac{n}{2}-A_1}_{g^n}(\phi)]$ that consists of $\vec{\xi}$-contractions of $\vec{\xi}$-length $\frac{n}{2}$ for which the decreasing rearrangement of the list $\nu_1,\dots,\nu_{\frac{n}{2}-A_1}$ is $(\Delta_1+2,2,\dots,2,1,\dots,1)$ (we are writing the number 2 Γ_1-1 times and 1 Δ_1 times. Then, by Lemma 3, we have that:

$$Shad_\alpha[I^{\frac{n}{2}-A_1}_{g^n}(\phi)]=0 \tag{69}$$

Now, we consider the sublinear combination $Shad_{\alpha,\beta}[I^{\frac{n}{2}-A_1}_{g^n}(\phi)]$ in $Shad_\alpha[I^{\frac{n}{2}-A_1}_{g^n}(\phi)]$ where there are no factors with internal contractions (in particular there are no factors ∇^pRic or ∇^mR_{ijkl} with internal contractions). Then, since the number of internal contractions remains invariant under the permutations of definition 7 in [2], modulo introducing $\vec{\xi}$-contractions of $\vec{\xi}$-length $\geq\frac{n}{2}+1$, we will have that modulo $\vec{\xi}$-contractions of $\vec{\xi}$-length $\geq\frac{n}{2}+1$:

$$Shad_{\alpha,\beta}[I^{\frac{n}{2}-A_1}_{g^n}(\phi)]=0 \tag{70}$$

Moreover, we define $Shad_{\alpha,\beta,\gamma}[I^{\frac{n}{2}-A_1}_{g^n}(\phi)]$ to stand for the sublinear combination in $Shad_{\alpha,\beta}[I^{\frac{n}{2}-A_1}_{g^n}(\phi)]$ where the Δ_1 factors $\nabla\phi$ are all contracting against the one factor $\nabla^{\Delta_1+2}\phi$. We observe that the number of factors $\nabla\phi$ that contract against the factor $\nabla^{\Delta_1+2}\phi$ remains invariant under the permutations allowed by definition 7 in [2], modulo introducing $\vec{\xi}$-contractions of $\vec{\xi}$-length $\geq \frac{n}{2}+1$. Hence, we have that modulo $\vec{\xi}$-contractions of $\vec{\xi}$-length $\geq \frac{n}{2}+1$:

$$Shad_{\alpha,\beta,\gamma}[I^{\frac{n}{2}-A_1}_{g^n}(\phi)] = 0 \tag{71}$$

Finally, we define $Shad_{\alpha,\beta,\gamma,\delta}[I^{\frac{n}{2}-A_1}_{g^n}(\phi)]$ to stand for the sublinear combination in $Shad_{\alpha,\beta,\gamma}[I^{\frac{n}{2}-A_1}_{g^n}(\phi)]$ that consists of the $\vec{\xi}$-contractions with X_1 factors $\nabla\vec{\xi}$ and no more factors of the form $S\nabla^u\vec{\xi}$ and, in addition, with C_1 factors $|\vec{\xi}|^2$. Since both the number of factors $S\nabla^p\vec{\xi}$ $(p \geq 1)$ and the number of such factors for which $p=1$, and also the number of factors $|\vec{\xi}|^2$ is invariant under the permutations of definition 7 in [2], we have that modulo $\vec{\xi}$-contractions of $\vec{\xi}$-length $\geq \frac{n}{2}+1$:

$$Shad_{\alpha,\beta,\gamma,\delta}[I^{\frac{n}{2}-A_1}_{g^n}(\phi)] = 0 \tag{72}$$

Now, we observe that $Shad_{\alpha,\beta,\gamma,\delta}[I^{\frac{n}{2}-A_1}_{g^n}(\phi)]$ indeed consists of $\vec{\xi}$-contractions of the form (66). This follows just because we are considering $\vec{\xi}$-length $\frac{n}{2}$ and weight $-n$. Hence, we must have $A_1-X_1-C_1$ factors $\nabla^m R_{ijkl}$ with no internal contractions. But since the $\vec{\xi}$-contractions in the form (66) have indeed weight $-n$, it follows that any $\vec{\xi}$-contraction with the restrictions above and with at least one factor $\nabla^m R_{ijkl}, m>0$ cannot have weight $-n$.

Now, for each complete contraction $C_{g^n}(\phi)$ in $I^{\frac{n}{2}-A_1}_{g^n}(\phi)$, we denote by $Tail^{Shad}_o[C_{g^n}(\phi)]$ the sublinear combination of $\vec{\xi}$-contractions in the form (66) in $Tail^{Shad}[C_{g^n}(\phi)]$. This notation extends to linear combinations.

Now, if we write $I^{\frac{n}{2}-A_1}_{g^n}(\phi)$ out as in (65), we claim that for each $C^u_{g^n}(\phi)$, where u is subsequent to the critical list, we have that modulo $\vec{\xi}$-contractions of $\vec{\xi}$-length $\geq \frac{n}{2}+1$:

$$Tail^{Shad}_o[C^u_{g^n}(\phi)] = 0 \tag{73}$$

We will prove this below. For now, we note how we can then determine our desired sublinear combination $\Sigma_{u\in U^{crit}} a_u C^u_{g^n}(\phi)$. Initially we observe that if we can show (73), we will then be able to determine the sublinear combination $\Sigma_{u\in U^{crit}} a_u Tail^{Shad}_o[C^u_{g^n}(\phi)]$ from equation (67). We then also claim that for each $u \in U^{crit}$, the sublinear combination $Tail^{Shad}_o[C^u_{g^n}(\phi)]$ is obtained from $C^u_{g^n}(\phi)$ by performing the following algorithm: We replace each factor R by $-|\vec{\xi}|^2$, each factor Ric_{ij} by $-\nabla_i\vec{\xi}_j$ and each factor $\Delta\phi$ by $\vec{\xi}^i\nabla_i\phi$ (in N-cancelled notation). We then integrate by parts the Δ_1 factors $\vec{\xi}$ that contract against factors $\nabla\phi$ and make each ∇_i that arises thus hit the same factor $\nabla^2\phi$.

This follows just by the iterative integration by parts procedure, and the same arguments as above. Since we have determined $\Sigma_{u\in U^{crit}} a_u Tail^{Shad}_o[C^u_{g^n}(\phi)]$, then by replacing each expression $|\vec{\xi}|^2$ by R, each expression $\nabla_i\vec{\xi}$ by $-Ric_{ij}$ and each expression $\nabla^{\Delta_1}_{s_1\dots s_{\Delta_1}}(\nabla^2_{f_1g_1})\phi\otimes\dots\otimes\nabla^2_{f_{\frac{n}{2}-A_1-\Delta_1}g_{\frac{n}{2}-A_1-\Delta_1}}\phi\otimes(\nabla)^{s_1}\phi\otimes$

$\dots \otimes (\nabla)^{s_{\Delta_1}}\phi)$ by $(\nabla^2_{f_1 g_1}\phi \otimes \dots \otimes \nabla_{f_{\frac{n}{2}-A_1-\Delta_1} g_{\frac{n}{2}-A_1-\Delta_1}}\phi)(\Delta\phi)^{\Delta_1}$, we have determined the sublinear combination $\Sigma_{u\in U^{crit}} a_u C^u_{g^n}(\phi)$. Moreover, we see that by construction, the pattern of those particular contractions between indices in factors $R_{ijkl}, Ric_{ij}, \nabla^2\phi$ is preserved.

So, matters are reduced to showing that for each $C^u_{g^n}(\phi)$ where u is subsequent to the critical character, we must have that $Tail^{Shad}_o[C^u_{g^n}(\phi)] = 0$. Firstly, we observe that we may restrict attention to the descendants of $C^u_{g^n}(\phi)$ that do not have internal contractions. This follows by the same reasoning as in the previous case. Then, we observe that if $C^u_{g^n}(\phi)$ has $\Delta < \Delta_1$ factors $\Delta\phi$, then each $\vec{\xi}$-contraction of length $\frac{n}{2}$ in $Tail^{Shad}[C^u_{g^n}(\phi)]$ will have less than Δ_1 factors $\nabla\phi$. Similarly, if $C^u_{g^n}(\phi)$ has less than C_1 factors R then each complete contraction of $\vec{\xi}$-length $\frac{n}{2}$ in $Tail^{Shad}[C^u_{g^n}(\phi)]$ will have less than C_1 expressions $|\vec{\xi}|^2$. Finally, if $C^u_{g^n}(\phi)$ has Δ_1 factors $\Delta\phi$, C_1 factors R and less than X_1 factors Ric_{ij}, then each $\vec{\xi}$-contraction in $Tail^{Shad}[C^u_{g^n}(\phi)]$ will either have less than X_1 factors $\nabla_i\vec{\xi}$ or less than C_1 expressions $|\vec{\xi}|^2$. Thus we have shown our claim.

The second case, where $\Delta_1 = \frac{n}{2} - A_1$ and $X_1 > 0$. We again consider the shadow divergence formula for $I^{\frac{n}{2}-A_1}_{g^n}(\phi)$, and we focus on the sublinear combination of $\vec{\xi}$-contractions in the form:

$$contr(R_{i_1 j_1 k_1 l_1} \otimes \dots \otimes R_{i_{A_1-X_1-C_1} j_{A_1-X_1-C_1} k_{A_1-X_1-C_1} l_{A_1-X_1-C_1}} \otimes S\nabla^{\frac{n}{2}-A_1+1}_{s_1 \dots s_{\frac{n}{2}-A_1} a_1}\vec{\xi}_{b_1} \otimes \nabla_{a_2}\vec{\xi}_{b_2} \otimes \dots \otimes \nabla_{a_{X_1}}\vec{\xi}_{b_{X_1}} \otimes (|\vec{\xi}|^2)^{C_1} \otimes (\nabla)^{s_1}\phi \otimes \dots \otimes (\nabla)^{s_{\Delta_1}}\phi) \tag{74}$$

We denote the above sublinear combination by $Shad_+[I^{\frac{n}{2}-A_1}_{g^n}(\phi)]$. Since the shadow divergence formula holds formally, by an analogous argument as for the previous case, it follows that:

$$Shad_+[I^{\frac{n}{2}-A_1}_{g^n}(\phi)] = 0 \tag{75}$$

For each complete contraction $C_{g^n}(\phi)$ in $I^{\frac{n}{2}-A_1}_{g^n}(\phi)$, we denote by $Tail^{Shad}_o[C^u_{g^n}(\phi)]$ the sublinear combination of $\vec{\xi}$-contractions in the form (74) in $Tail^{Shad}[C_{g^n}(\phi)]$. (This is not the same as the previous $Tail^{Shad}[C_{g^n}(\phi)]$).

Now, by a similar reasoning as for the previous case, we observe that for each $C^u_{g^n}(\phi)$ that is subsequent to the critical character we have $Tail^{Shad}_o[C^u_{g^n}(\phi)] = 0$. This follows because if $C^u_{g^n}(\phi)$ has either less than Δ_1 factors $\Delta\phi$, or Δ_1 such factors and less than C_1 factors R or C_1 such factors and less than X_1 factors Ric. In those cases, we respectively have that each $\vec{\xi}$-contraction in $Tail[C^u_{g^n}(\phi)]$ will have less than Δ_1 factors $\nabla\phi$ or less than C_1 factors $|\vec{\xi}|^2$ or less than X_1 factors $S\nabla^p\vec{\xi}$. Hence, using (75), we determine the sublinear combination $\Sigma_{u\in U^{crit}} a_u Tail_o^{Shad}[C^u_{g^n}(\phi)]$.

We now claim that for each $C^u_{g^n}(\phi), u \in U^{crit}$, the sublinear combination $Tail_o^{Shad}[C^u_{g^n}(\phi)]$ arises as follows: We initially replace each of the C_1 factors R by $|\vec{\xi}|^2$, each of the X_1 factors Ric_{ij} by $-\nabla_i\vec{\xi}_j$ and each of the $\frac{n}{2}-A_1$ factors $\Delta\phi$ by $\nabla^i\phi\vec{\xi}_i$ (we are using N-cancelled notation). We then integrate by parts the $\frac{n}{2}-A_1$ factors $\vec{\xi}$ that contract against a factor $\nabla\phi$ and make the derivatives ∇^i hit the same one factor $\nabla_i\vec{\xi}_j$ and replace $\nabla^{\frac{n}{2}-A_1}_{i_1\dots i_{\frac{n}{2}-A_1}}\nabla_i\vec{\xi}_j$ by $S\nabla^{\frac{n}{2}-A_1}_{i_1\dots i_{\frac{n}{2}-A_1}i}\vec{\xi}_j$. This follows by the iterative integrations by parts procedure, as in the previous case.

Therefore, once we have determined $\Sigma_{u\in U^{crit}} a_u Tail_o^{Shad}[C^u_{g^n}(\phi)]$, we can determine $\Sigma_{u\in U^{crit}} a_u C^u_{g^n}(\phi)$ as follows: We replace each factor $|\vec{\xi}|^2$ by R, each factor $\nabla_i\vec{\xi}_j$ by $-Ric_{ij}$ and each expression $S\nabla^{\frac{n}{2}-A_1+1}_{s_1\dots s_{\frac{n}{2}-A_1}a_1}\vec{\xi}_{b_1}(\nabla)^{s_1}\phi\otimes\dots\otimes(\nabla)^{s_{\Delta_1}}\phi$ by $Ric_{a_1b_1}(\Delta\phi)^{\frac{n}{2}-A_1}$. We then determine the sublinear combination $\Sigma_{u\in U^{crit}} a_u C^u_{g^n}(\phi)$.

The third case.

Finally, we have to consider the third case. We now consider $I^{\frac{n}{2}-A_1}_{g^n}(\phi)$ and distinguish the two subcases $C_1 = 0$ or $C_1 > 0$.

The first subcase $C_1 = 0$. Modulo complete contractions of length $\geq \frac{n}{2}+1$, we write out $I^{\frac{n}{2}}_{g^n}(\phi)$ in the form:

$$I^{\frac{n}{2}}_{g^n}(\phi) = \Sigma_{g\in G} a_g C^g_{g^n}(\phi) + \Sigma_{u\in U^{crit}} a_u C^u_{g^n}(\phi) + \Sigma_{u\in U^{subs}} a_u C^u_{g^n}(\phi) \tag{76}$$

where $\Sigma_{g\in G} a_g C^g_{g^n}(\phi)$ stands for the known sublinear combination in $I^{\frac{n}{2}}_{g^n}(\phi)$ (this now includes a part of $\Sigma_{u\in U} a_u C^u_{g^n}(\phi)$). $\Sigma_{u\in U^{crit}} a_u C^u_{g^n}(\phi)$ stands for the sublinear combination of compete contractions indexed in the critical list, U^{crit}. Finally, $\Sigma_{u\in U^{subs}} a_u C^u_{g^n}(\phi)$ stands for the sublinear combination of complete contractions $C^u_{g^n}(\phi)$ that are subsequent to the critical list.

We focus on the super divergence formula for $I^{\frac{n}{2}}_{g^n}(\phi)$. We pick out the sublinear combination of complete contractions in the form:

$$contr(\nabla^{\Delta_1}_{r_1\dots r_{\Delta_1}} R_{i_1j_1k_1l_1}\otimes R_{i_2j_2k_2l_2}\otimes\dots\otimes R_{i_{A_1}j_{A_1}k_{A_1}l_{A_1}}\otimes(\nabla)^{s_1}\phi\otimes\dots\otimes(\nabla)^{s_{\frac{n}{2}-A_1}}\phi) \tag{77}$$

where each of the factors $\nabla\phi$ contracts against an index in the factor $\nabla^{\Delta_1}R_{ijkl}$.

We denote the corresponding sublinear combination of complete contractions in $supdiv[I^{\frac{n}{2}-A_1}_{g^n}(\phi)]$ by $supdiv_+[I^{\frac{n}{2}-A_1}_{g^n}(\phi)]$. Since the super divergence formula holds formally, it follows that:

$$supdiv_+[I^{\frac{n}{2}-A_1}_{g^n}(\phi)] = 0$$

modulo complete contractions of length $\geq \frac{n}{2}+1$. Now, for each $C_{g^n}(\phi)$ in $I_{g^n}^{\frac{n}{2}-A_1}(\phi)$, we denote by $Tail_+[C_{g^n}(\phi)]$ the sublinear combination in $Tail[C_{g^n}(\phi)]$ that consists of complete contractions in the form (77).

We then again observe that for each u that is subsequent to the critical list, we have $Tail_+[C^u_{g^n}(\phi)]=0$. This follows since if $C^u_{g^n}(\phi)$ is subsequent to the critical list it must have less than $\frac{n}{2}-A_1$ factors $\Delta\phi$, hence any complete contraction of length $\frac{n}{2}$ in $Tail[C^u_{g^n}(\phi)$ must have less than $\frac{n}{2}-A_1$ factors $\nabla\phi$. On the other hand, for each $u\in U^{crit}$ we have that $Tail_+[C^u_{g^n}(\phi)]$ arises from $C^u_{g^n}(\phi)$ as follows: We replace each of the factors $\Delta\phi$ by $\nabla_i\phi\vec{\xi^i}$ and then integrate by parts the $\frac{n}{2}-A_1$ factors $\vec{\xi}$ and make each of them hit the same factor R_{ijkl} (there are A_1 choices of the factor R_{ijkl} that we may pick). The sublinear combination that arises thus is $Tail_+[C^u_{g^n}(\phi)]$. In fact, we observe that if $C^u_{g^n}(\phi)$ is of the form:

$$contr(R_{i_1j_1k_1l_1}\otimes\cdots\otimes R_{i_{A_1}j_{A_1}k_{A_1}l_{A_1}}\otimes(\Delta\phi)^{\frac{n}{2}-A_1}) \tag{78}$$

Then $Tail_+[C^u_{g^n}(\phi)]$ can be written as a sum of A_1 complete contractions in the form:

$$(-1)^{\frac{n}{2}-A_1}contr(\nabla^{\frac{n}{2}-A_1}_{i_1\dots i_{\frac{n}{2}-A_1}}R_{ijkl}\otimes\cdots\otimes R_{i'j'k'l'}\otimes(\nabla)^{i_1}\phi\otimes\cdots\otimes(\nabla)^{i_{\frac{n}{2}-A_1}}\phi) \tag{79}$$

where the h^{th} term in the sum arises from $C^u_{g^n}(\phi)$ by replacing all the factors $\Delta\phi$ by a factor $\nabla_{a_j}\phi$ $(1\leq j\leq\frac{n}{2}-A_1)$ and then hitting the h^{th} factor R_{ijkl} in $C^u_{g^n}(\phi)$ by $\frac{n}{2}-A_1$ derivatives $(\nabla)^{a_j}$. In order to facilitate our work further down, we will write out:

$$Tail_+[C^u_{g^n}(\phi)]=\Sigma_{h=1}^{A_1}C^{u,h}_{g^n}(\phi) \tag{80}$$

where $C^{u,h}_{g^n}(\phi)$ stands for the h^{th} complete contraction explained above. Given the form (78) of $C^u_{g^n}(\phi)$, we have that $C^{u,h}_{g^n}(\phi)$ will be in the form:

$$contr(R_{i_1j_1k_1l_1}\otimes\cdots\otimes\nabla_{i_1\dots i_{\Delta_1}}R_{i_hj_hk_hl_h}\otimes\cdots\otimes R_{i_{A_1}j_{A_1}k_{A_1}l_{A_1}}\otimes(\nabla)^{s_1}\phi\otimes\cdots\otimes(\nabla)^{s_{\Delta_1}}\phi) \tag{81}$$

Now, for each $u\in U^{crit}$, we denote by $C^u(g^n)$ the complete contraction of weight $-2A_1$:

$$contr(R_{i_1j_1k_1l_1}\otimes\cdots\otimes R_{i_{A_1}j_{A_1}k_{A_1}l_{A_1}})$$

We then claim that we can determine the linear combination $\Sigma_{u\in U^{crit}}a_uC^u(g^n)$. Given the form (78) of each $C^u_{g^n}(\phi), u\in U^{crit}$, that would then imply that we can determine the sublinear combination $\Sigma_{u\in U^{crit}}a_uC^u_{g^n}(\phi)$, and the proof of our third case for the subcase $C_1=0$ would be complete. In order to determine $\Sigma_{u\in U^{crit}}a_uC^u(g^n)]$, we do the following:

We may re-express $supdiv_+[I^{\frac{n}{2}-A_1}_{g^n}(\phi)]$ in the form:

$$\Sigma_{u\in U^{crit}}a_u Tail_+[C^u_{g^n}(\phi)]+\Sigma_{g\in G}a_g Tail_+[C^g_{g^n}(\phi)]=0 \quad (82)$$

modulo complete contractions of length $\geq \frac{n}{2}+1$. Here each $Tail_+[C^g_{g^n}(\phi)]$ consists of complete contractions in the form (77) and since the sublinear combination $\Sigma_{g\in G}a_g C^g_{g^n}(\phi)$ is known, we have that the sublinear combination $\Sigma_{g\in G}a_g Tail_+[C^g_{g^n}(\phi)]$ is known. Alternatively, in our new notation using (80):

$$\Sigma_{u\in U^{crit}}a_u \Sigma_{h=1}^{A_1}C^{u,h}_{g^n}(\phi)+\Sigma_{g\in G}a_g Tail_+[C^g_{g^n}(\phi)]=0 \quad (83)$$

modulo complete contractions of length $\geq \frac{n}{2}+1$. We will then determine the sublinear combination $\Sigma_{u\in U^{crit}}a_u C^u(g^n)$ by a trick:

Initially, we polarize the $\frac{n}{2}-A_1$ functions ϕ in the above equation. We denote by $C^{u,h}_{g^n}(\phi_1,\dots,\phi_{\Delta_1})$ the complete contraction:

$$contr(R_{i_1j_1k_1l_1}\otimes\dots\otimes\nabla_{i_1\dots i_{\Delta_1}}R_{i_hj_hk_hl_h}\otimes\dots\otimes R_{i_{A_1}j_{A_1}k_{A_1}l_{A_1}}\otimes(\nabla)^{i_1}\phi_1\otimes\dots\otimes(\nabla)^{i_{\Delta_1}}\phi_{\Delta_1}) \quad (84)$$

We also denote by $\Sigma_{g\in G}a_g Tail_+[C^g_{g^n}(\phi_1,\dots,\phi_{\Delta_1})]$ the sublinear combination of complete contractions that arises from $\Sigma_{g\in G}a_g Tail_+[C^g_{g^n}(\phi)]$ by polarizing the Δ_1 functions ϕ. It will be a linear combination of complete contractions in the form:

$$contr(\nabla^{\Delta_1}_{t_1\dots t_{\Delta_1}}R_{i_1j_1k_1l_1}\otimes R_{i_2j_2k_2l_2}\otimes\dots\otimes R_{i_{A_1}j_{A_1}k_{A_1}l_{A_1}}\otimes(\nabla)^{i_1}\phi_1\otimes\dots\otimes(\nabla)^{i_{\frac{n}{2}-A_1}}\phi_{\Delta_1}) \quad (85)$$

where each $\nabla\phi_h$ contracts against the same factor $\nabla^{\Delta_1}_{i_1\dots i_{\Delta_1}}R_{i_1j_1k_1l_1}$. Again, since $\Sigma_{g\in G}a_g C^g_{g^n}(\phi_1,\dots,\phi_{\Delta_1})$ arises from $\Sigma_{g\in G}a_g C^g_{g^n}(\phi)$ by polarization, we have that the sublinear combination $\Sigma_{g\in G}a_g C^g_{g^n}(\phi_1,\dots,\phi_{\Delta_1})$ is known. Therefore, from (83) we derive an equation modulo complete contractions of length $\geq \frac{n}{2}+1$:

$$\Sigma_{u\in U^{crit}}a_u\Sigma_{h=1}^{A_1}C^{u,h}_{g^n}(\phi_1,\dots,\phi_{\Delta_1})]+\Sigma_{g\in G}a_g Tail_+[C^g_{g^n}(\phi_1,\dots,\phi_{\Delta_1})]=0 \quad (86)$$

Definition 1 *For each $0\leq\kappa\leq\Delta_1$, we define $C^{u,h}_{g^n}(\phi_{\kappa+1},\dots,\phi_{\Delta_1})$ to stand for the complete contraction:*

$$contr(R_{i_1j_1k_1l_1}\otimes\dots\otimes\nabla_{i_{\kappa+1}\dots i_{\Delta_1}}R_{i_hj_hk_hl_h}\otimes\dots\otimes R_{i_{A_1}j_{A_1}k_{A_1}l_{A_1}}\otimes(\nabla)^{i_{\kappa+1}}\phi_{\kappa+1}$$
$$\otimes\dots\otimes(\nabla)^{i_{\Delta_1}}\phi_{\Delta_1}) \quad (87)$$

It arises from $C^{u,h}_{g^n}(\phi_1,\dots,\phi_{\Delta_1})$ by erasing the factors $\nabla\phi_h, h\leq\kappa$ and also erasing the indices that they contract against in the factor $\nabla^{\Delta_1}R_{i_hj_hk_hl_h}$. We observe that for $\kappa=0$, our notation is consistent. We also have for $\kappa+1=\Delta_1$, we obtain $C^u(g^n)$. We note that by construction $C^{u,h}_{g^n}(\phi_1,\dots,\phi_{\Delta_1})$ has length $\frac{n}{2}-\kappa$.

We now consider complete contractions of the form:

$$contr(\nabla^{\Delta_1-\kappa}_{r_1\dots r_{\Delta_1-\kappa}} R_{i_1j_1k_1l_1}\otimes R_{i_2j_2k_2l_2}\otimes\dots\otimes R_{i_{A_1}j_{A_1}k_{A_1}l_{A_1}}\otimes(\nabla)^{i_{k+1}}\phi_{\kappa+1}\otimes\dots \\ \otimes(\nabla)^{i_{\frac{n}{2}-A_1}}\phi_{\Delta_1}) \tag{88}$$

where each of the factors $\nabla\phi_h$ contracts against an index in the factor $\nabla^{\Delta_1-\kappa}R_{ijkl}$. We observe that up to switching the position of the factor $\nabla^{\Delta_1-\kappa-1}R_{ijkl}$ and a factor $R_{i'j'k'l'}$, the complete contractions $C^{u,h}_{g^n}(\phi_{\kappa+1},\dots,\phi_{\Delta_1})$ are in the form (88) above.

We now let $\Sigma_{g\in G^\kappa}a_g C^g_{g^n}(\phi_{\kappa+1},\dots,\phi_{\Delta_1})$ stand for a generic *known* linear combination of complete contractions in the form (88).

Our claim is then the following:

Lemma 7 *We claim that for any $\kappa, 0\le\kappa\le\Delta_1$, we will have that modulo complete contractions of length $\ge\frac{n}{2}-\kappa+1$:*

$$\Sigma_{u\in U^{crit}}a_u\Sigma^{A_1}_{h=1}C^{u,h}_{g^n}(\phi_{\kappa+1},\dots,\phi_{\Delta_1})+\Sigma_{g\in G^\kappa}a_g C^g_{g^n}(\phi_{\kappa+1},\dots,\phi_{\Delta_1})=0 \tag{89}$$

Clearly if we can show the above, then using the case $\kappa=\Delta_1$, we will then have shown our third case above in the first subcase. The equation holds exactly because terms of greater length have the wrong weight.

Proof: We will prove the above by an induction. We assume that we know our Lemma for $\kappa=k$ and we will show it for $\kappa=k+1$, where $k\le\Delta_1$.

We write out our inductive hypothesis:

$$L_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})=\Sigma_{u\in U^{crit}}a_u\Sigma^{A_1}_{h=1}C^{u,h}_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1}) \\ +\Sigma_{g\in G^k}a_g C^g_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})=\Sigma_{y\in Y}a_y C^y_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1}) \tag{90}$$

where each $C^y_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})$ has length $\ge\frac{n}{2}-k+1$.

For each complete contraction $C_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})$ of weight $-n+2k$ we define, for the purposes of this proof:

$$Image^1_{\phi'}[C_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})]=\partial_\lambda|_{\lambda=0}[e^{\lambda(n-2k)\phi'}C_{e^{2\lambda\phi'}g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})] \tag{91}$$

Now, by our inductive hypothesis, we deduce that:

$$Image^1_{\phi'}\{\Sigma_{u\in U^{crit}}a_u\Sigma^{A_1}_{h=1}C^{u,h}_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})\}+ \\ Image^1_{\phi'}\{\Sigma_{g\in G^k}a_g C^g_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})\}=Image^1_{\phi'}\{\Sigma_{y\in Y}a_y C^y_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})\} \tag{92}$$

We make a note on how the operation $Image^1_{\phi'}$ acts: Consider any complete contraction $C_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})$ of weight $-n+2k$. Then, $Image^1_{\phi'}[C_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})]$ is determined as follows: We arbitrarily pick out one factor T_{g^n} in $C_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})$ and we make all its indices free. We thus have a tensor $T^{g^n}_{i_1\dots i_h}$. Then, consider all the terms in $T^{e^{2\phi'}g^n}_{i_1\dots i_h}$ that are linear in ϕ' and involve at least one derivative of ϕ'. We arbitrarily replace T_{g^n} in $C_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})$ by one of those terms, we leave all the other factors unaltered, and perform the same particular contractions as for $C_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})$. Adding over all these arbitrary substitutions, we obtain $Image^1_{\phi'}[C_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})]$.

Now, we restrict our attention to complete contractions $C_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})$ in the form (88) and we wish to understand which complete contractions in $Image^1_{\phi'}[C_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})]$ are in the form:

$$contr(\nabla^{\Delta_1-\kappa-1}R_{i_1j_1k_1l_1}\otimes R_{i_2j_2k_2l_2}\otimes R_{i_{A_1}j_{A_1}k_{A_1}l_{A_1}}\otimes\nabla\phi_{k+2}\otimes\nabla\phi_{\Delta_1}\otimes\dots\otimes(\nabla)^h\phi' \otimes\nabla_h\phi_{k+1}) \quad (93)$$

In the above complete contraction, the length is $\frac{n}{2}-k+1$ and each of the factors $\nabla\phi_h, h\geq k+2$ contracts against the factor $\nabla^{\Delta_1-k-1}R_{ijkl}$ and the two factors $\nabla\phi_{k+1},\nabla\phi'$ contract between themselves. We will call such contractions *targets*. We denote their sublinear combination in each $Image^1_{\phi'}[C_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})]$ by $Image^{1,targ}_{\phi'}[C_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})]$.

Now, let us further analyze each $Image^1_{\phi'}[C_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})]$, where $C_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})$ is in the form (88). For each factor $T_f=\nabla^mR_{ijkl}$ ($m\geq 0$, $1\leq f\leq A_1$), we denote by $Full_{T_f}[C_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})]$ the sum of four complete contractions that arises from $C_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})$ by replacing the factor $T=\nabla^mR_{ijkl}$ by one of the linear expressions $\nabla^m(\nabla^2\phi'\otimes g)$ on the right hand side of (28) and then adding those four substitutions. It follows that each $Full_{T_f}[C_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})]$ is a sum of four complete contractions of length $\frac{n}{2}-k$, each in the form:

$$contr(\nabla^{m_1}R_{ijkl}\otimes\dots\otimes\nabla^{m_{A_1}-1}R_{i'j'k'l'}\otimes\nabla^r\phi'\otimes\nabla\phi_{k+1}\otimes\dots\otimes\nabla\phi_{\Delta_1}) \quad (94)$$

where $r\geq 2$, and each $m_u\geq 0$. This follows from the transformation law (28).

On the other hand, for each $C_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})$, we make note of the one factor ∇^mR_{ijkl} with $m>0$ and we call it *critical*. We let $LC^{crit}[C_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})]$ stand for the sublinear combination that arises in $Image^1_{\phi'}[C_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})]$ when we replace the critical factor by an expression $\nabla^hR_{ijkl}\nabla^b\phi'$ or $\nabla^hR_{ijkl}\nabla^b\phi' g_{ab}$, that arises either by virtue of the transformation law (29) or by virtue of the homogeneity of R_{ijkl} (see (28)).

Then, for each complete contraction $C_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})$ on the left hand side of (92) we have:

$$\begin{aligned} &Image^1_{\phi'}[C_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})] = \Sigma_{f=1}^{A_1} Full_{T_f}[C_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})] \\ &+ LC^{crit}[C_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})] \end{aligned} \tag{95}$$

We will now show that:

$$\begin{aligned} &\Sigma_{u\in U^{crit}} a_u \Sigma_{f=1}^{A_1} Full_{T_f}[C^{u,h}_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})] + \\ &\Sigma_{g\in G} a_g \Sigma_{f=1}^{A_1} Full_{T_f}[C^{g}_{g^n}(\phi_{k+2},\dots,\phi_{\Delta_1})] = \Sigma_{j\in J} a_j C^j_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1},\phi') \end{aligned} \tag{96}$$

where each $\Sigma_{j\in J} a_j C^j_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1},\phi')$ has length $\ge \frac{n}{2}-k+1$ and is not a target.

We see this as follows: Initially, we recall equation (92), where the left hand side can be explicitly written out by virtue of (95) and the right hand side consists of complete contractions of length $\ge \frac{n}{2}-k+1$. This follows from (28) and (29). Therefore, recalling that each $LC^{crit}[C_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})$ in (95) consists of complete contractions of length $\frac{n}{2}-k+1$, we have:

$$\begin{aligned} &\Sigma_{u\in U^{crit}} a_u \Sigma_{h=1}^{A_1} \Sigma_{f=1}^{A_1} Full_{T_f}[C^{u,h}_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})] \\ &+ \Sigma_{g\in G^k} a_g \Sigma_{f=1}^{A_1} Full_{T_f}[C^{g}_{g^n}(\phi_{k+2},\dots,\phi_{\Delta_1})] = 0 \end{aligned} \tag{97}$$

modulo complete contractions of length $\ge \frac{n}{2}-k+1$.

Now, the above holds formally. Hence, there is a sequence of permutations among the indices of the factors in the left hand side of the above with which we can make the left hand side of the above formally zero, modulo introducing complete contractions of length $\ge \frac{n}{2}-k+1$. We want to keep track of the correction terms that arise. We see that the correction terms can only arise by applying the identity $[\nabla_A\nabla_B - \nabla_B\nabla_A]X_C = R_{ABCD}X^D$. But we see that if we apply this identity to a factor $\nabla^m R_{ijkl}$, we introduce a correction term of length $\frac{n}{2}-k+1$ *which will have a factor* $\nabla^r\phi'$, $r\ge 2$. This is true because each expression consists of complete contractions in the form (94), so there is such a factor to begin with. Hence, we do not obtain a target in this way. On the other hand, if we apply the identity $[\nabla_A\nabla_B - \nabla_B\nabla_A]X_C = R_{ABCD}X^D$ to the factor $\nabla^r\phi'$, $r\ge 2$, we will obtain a correction term which will either have a factor $\nabla^u\phi'$, $u\ge 2$ *or* a factor $\nabla\phi'$ *which contracts against a factor* $\nabla^t R_{ijkl}$. Therefore, we do not obtain a targets in this way either. We have shown (96).

Our next claim is:

Claim A: For each $u \in U^{crit}$, $1 \le h \le A_1$:

$$\begin{aligned}
&LC^{crit}[C^{u,h}_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})] = \\
&(-2-(\Delta_1-k-1))C^{u,h}_{g^n}(\phi_{k+2},\dots,\phi_{\Delta_1})(\nabla)^h\phi'\nabla_h\phi_{k+1} \\
&+\Sigma_{j\in J}a_jC^j_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})
\end{aligned} \tag{98}$$

where the linear combination $\Sigma_{j\in J}a_jC^j_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1},\phi')$ is a generic linear combination of complete contractions of length $\frac{n}{2}-k+1$ that are not targets.

We show claim A as follows: For each complete contraction $C_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})$ appearing on the left hand side of (90), we have defined $LC^{crit}[C_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})]$. Now, we pay special attention to the one index i_{k+1} in the critical factor that is contracting against the factor $\nabla\phi_{k+1}$. Let $LC^{crit,\alpha}[C_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})]$ be the sublinear combination that arises in $LC^{crit}[C_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})]$ when we replace the critical factor $\nabla^{m-k}_{r_{k+1}\dots r_m}R_{ijkl}$ by an expression $\nabla_{r_{k+1}}\phi'\nabla^{m-k-1}R_{ijkl}$. (Note that the index r_{k+1} is the one that contracted against the factor $\nabla\phi_{k+1}$ in $C_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})$). We denote by $LC^{crit,\beta}[C_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})]$ the sublinear combination that arises in $LC^{crit}[C_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})]$ when we replace the critical factor in any other way.

Hence, $LC^{crit,\beta}[C_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})]$ arises by replacing the critical factor by an expression in either the form $\nabla^h\phi'\nabla^uR_{ijkl}$, $\nabla^h\phi'\nabla^uR_{ijkl}g_{ab}$ with $h\ge 2$ or of the form $\nabla_\alpha\phi'\nabla^uR_{ijkl}$, $\nabla_\alpha\phi'\nabla^uR_{ijkl}g_{ab}$ where the index α is not the index r_{k+1} that contracts against $\nabla\phi_{k+1}$.

We observe that the sublinear combinations $LC^{crit,\alpha}[C^{u,h}_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})]$, $LC^{crit,\alpha}[C^g_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})]$ consist of targets, whereas the sublinear combinations $LC^{crit,\beta}[C^{u,h}_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})]$, $LC^{crit,\beta}[C^g_{g^n}(\phi_{1+1},\dots,\phi_{\Delta_1})]$ contain no targets.

Therefore, in view of the above, in order to show Claim A, we only have to show that for each $u\in U^{crit}$ and each $1\le h\le A_1$, we have that:

$$\begin{aligned}
&LC^{crit,\alpha}[C^{u,h}_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})] = \\
&(-2-(\Delta_1-k-1))\cdot C^{u,h}_{g^n}(\phi_{k+2},\dots,\phi_{\Delta_1})(\nabla)^h\phi_1\nabla_h\phi'
\end{aligned} \tag{99}$$

Hence, we only have to show that the sublinear combination of expressions in $Image^1_{\phi'}[\nabla^{\Delta_1-k}_{r_{k+1}\dots r_{\Delta_1}}R_{ijkl}]$ that are in the form $\nabla_{r_{k+1}}\phi'\nabla^{\Delta_1-k}_{r_{k+2}\dots r_{\Delta_1-\kappa}}R_{ijkl}$ is precisely $(-2-(\Delta_1-k-1))\cdot\nabla_{r_{k+1}}\phi'\nabla^{\Delta_1-\kappa}_{r_{k+2}\dots r_{\Delta_1-\kappa}}R_{ijkl}$. But this is only a matter of applying (29) to all the pairs (r_{k+1},r_a), $a\ge k+2$ and the pairs $(r_{k+1},i),\dots(r_{k+1},l)$ and also by taking into account the expression $2\nabla_{r_{k+1}}\phi'\nabla^{\Delta_1-k-1}_{r_{k+2}\dots r_{\Delta_1}}R_{ijkl}$ that arises by virtue of the homogeneity of the factor R_{ijkl}.

Combining the equations (92), (95), (96), (98) and (99) above, we have that:

$$\Sigma_{j\in J}a_jC^j_{g^n}(\phi_{k+1},\dots,\psi_{\Delta_1},\phi')+\Sigma_{u\in U^{crit}}a_u\Sigma_{h=1}^{A_1}LC^{crit,\alpha}[C^{u,h}_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})]$$
$$+\Sigma_{g\in G^k}a_gLC^{crit,\alpha}[C^g_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})]+\Sigma_{u\in U^{crit}}a_u\Sigma_{h=1}^{A_1}LC^{crit,\beta}[C^{u,h}_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})]$$
$$+\Sigma_{g\in G^k}a_gLC^{crit,\beta}[C^g_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})]=\Sigma_{z\in Z}a_zC^z_{g^n}(\phi_1,\dots,\phi_{\Delta_1},\phi') \tag{100}$$

where the sublinear combination $\Sigma_{z\in Z}a_zC^z_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1},\phi')$ stands for:

$$Image^1_{\phi'}[\Sigma_{y\in Y}a_yC^y_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})],$$

and hence each $C^z_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1},\phi')$ either has length $\ge\frac{n}{2}-\kappa+2$ or has length $\ge\frac{n}{2}-k+1$ but has a factor $\nabla^u\phi',u\ge 2$ (so it is not a target). Therefore, since (100) must hold formally, we deduce that, modulo complete contractions of length $\ge\frac{n}{2}-k+2$:

$$\Sigma_{u\in U^{crit}}a_u\Sigma_{h=1}^{A_1}LC^{crit,\alpha}[C^{u,h}_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})]$$
$$+\Sigma_{g\in G^k}a_gLC^{crit,\alpha}[C^g_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})]=0 \tag{101}$$

Now, since we are assuming that the linear combination $\Sigma_{g\in G^k}a_gC^g_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})$ is known, we deduce that the linear combination $\Sigma_{g\in G^k}a_gLC^{crit,\alpha}[C^g_{g^n}(\phi_{k+1},\dots,\phi_{\Delta_1})]$ is also known.

We have thus completed the proof of the third case if $C_1=0$. □

The subcase $C_1>0$:

The second subcase is almost entirely similar. We again write out $I^{\frac{n}{2}}_{g^n}(\phi)$ in the form (76). We write $C_1=\gamma$. We consider the Shadow divergence formula for $I^{\frac{n}{2}-A_1}_{g^n}(\phi)$ and we focus on the sublinear combination $Shad_+[I^{\frac{n}{2}-A_1}_{g^n}(\phi)]$ in $Shad[I^{\frac{n}{2}-A_1}_{g^n}(\phi)]$ which consists of $\vec{\xi}$-contractions in the form:

$$contr(\nabla^{\Delta_1}_{t_1\dots t_{\Delta_1}}R_{i_1j_1k_1l_1}\otimes\dots\otimes R_{i_2j_2k_2l_2}\otimes R_{i_{A_1-\gamma}j_{A_1-\gamma}k_{A_1-\gamma}l_{A_1-\gamma}})$$
$$\otimes(|\vec{\xi}|^2)^\gamma\otimes(\nabla)^{s_1}\phi\otimes\dots\otimes(\nabla)^{s_{\frac{n}{2}-A_1}}\phi) \tag{102}$$

where each a factor $\nabla\phi$ contracts against an index in the factor $\nabla^{\Delta_1}R_{ijkl}$.

As in all the previous cases, we have that:

$$Shad_+[I^{\frac{n}{2}-A_1}_{g^n}(\phi)]=0 \tag{103}$$

modulo $\vec{\xi}$-contractions of length $\ge\frac{n}{2}+1$, since the shadow divergence formula holds formally.

As before, for each $C^u_{g^n}(\phi)$ that is subsequent to the critical list, we have that $Tail^{Shad}_+[C^u_{g^n}(\phi)] = 0$. Hence, we have that:

$$\Sigma_{g\in G} a_g Shad_+[C^g_{g^n}(\phi)] + \Sigma_{u\in U^{crit}} a_u Shad_+[C^u_{g^n}(\phi)] = 0 \tag{104}$$

modulo $\vec{\xi}$-contractions of $\vec{\xi}$-length $\geq \frac{n}{2}+1$.

Moreover, for each $u \in U^{crit}$, where $C^u_{g^n}(\phi)$ is in the form:

$$contr(R_{i_1j_1k_1l_1} \otimes \dots \otimes R_{i_{A_1-\gamma}j_{A_1-\gamma}k_{A_1-\gamma}l_{A_1-\gamma}} \otimes R^\gamma \otimes \Delta\phi \otimes \dots \otimes \Delta\phi) \tag{105}$$

we have that $Tail^{Shad}_+[C^u_{g^n}(\phi)]$ can be written out as:

$$Tail^{Shad}_+[C^u_{g^n}(\phi)] = \Sigma^{A_1-\gamma}_{h=1} C^{u,h}_{g^n}(\phi) \tag{106}$$

where $C^{u,h}_{g^n}(\phi)$ is in the form:

$$contr(R_{i_1j_1k_1l_1} \otimes \dots \otimes \nabla^{i_1\dots i_{\frac{n}{2}-A_1}} R_{i_hj_hk_hl_h} \otimes \dots \otimes R_{i_{A_1-\gamma}j_{A_1-\gamma}k_{A_1-\gamma}l_{A_1-\gamma}} \\ \otimes (|\vec{\xi}|^2)^\gamma \otimes \nabla_{i_1}\phi \otimes \dots \otimes \nabla_{i_{\frac{n}{2}-A_1}}\phi) \tag{107}$$

We then define $C^u(g^n)$ to stand for the complete contraction:

$$contr(R_{i_1j_1k_1l_1} \otimes \dots \otimes R_{i_{A_1-\gamma}j_{A_1-\gamma}k_{A_1-\gamma}l_{A_1-\gamma}})$$

Hence, using the equation (104) and repeating the same argument as in the above case, we may determine the sublinear combination $\Sigma_{u\in U^{crit}} a_u C^u(g^n)$, and hence also the sublinear combination $\Sigma_{u\in U^{crit}} a_u C^u_{g^n}(\phi)$. We have completed the proof of Lemma 2. □

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