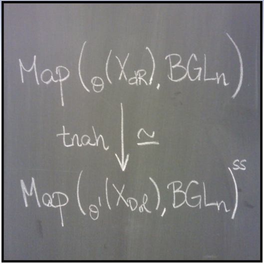

A twisted nonabelian Hodge correspondence

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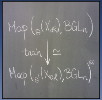


Outline

Connections and Higgs fields

X – smooth complex projective variety.

$\mathcal{E}, \mathcal{F}$ – holomorphic vector bundles of rank n on X.

Flat connections

$\mathbb{C}$-linear map $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X$ satisfying the Leibniz rule:

$$\nabla(fv) = v \otimes df + f \nabla v$$

Curvature $C(\nabla)$:

$$\mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes \Omega^1_X \xrightarrow{\nabla} \mathcal{E} \otimes \Omega^2_X$$

Flatness: $C(\nabla) = 0.$

Higgs fields

$\mathcal{O}_X$-linear map $\phi : \mathcal{F} \to \mathcal{F} \otimes \Omega^1_X.$

Curvature $C(\phi)$:

$$\mathcal{F} \xrightarrow{\phi} \mathcal{F} \otimes \Omega^1_X \xrightarrow{\phi} \mathcal{F} \otimes \Omega^2_X$$

Flatness: $C(\phi) = 0.$

Connections and Higgs fields

X – smooth complex projective variety.

$\mathcal{E}, \mathcal{F}$ – holomorphic vector bundles of rank n on X.

Flat λ-connections

$\mathbb{C}$-linear map $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X$
satisfying the λ-twisted Leibniz rule:

$$\nabla(fv) = \lambda v \otimes df + f\nabla v$$

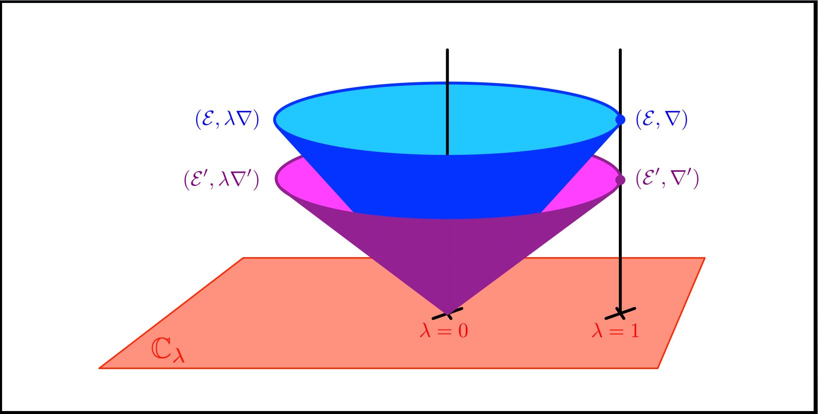
Curvature $C(\nabla)$:

$$\mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes \Omega^1_X \xrightarrow{\nabla} \mathcal{E} \otimes \Omega^2_X$$

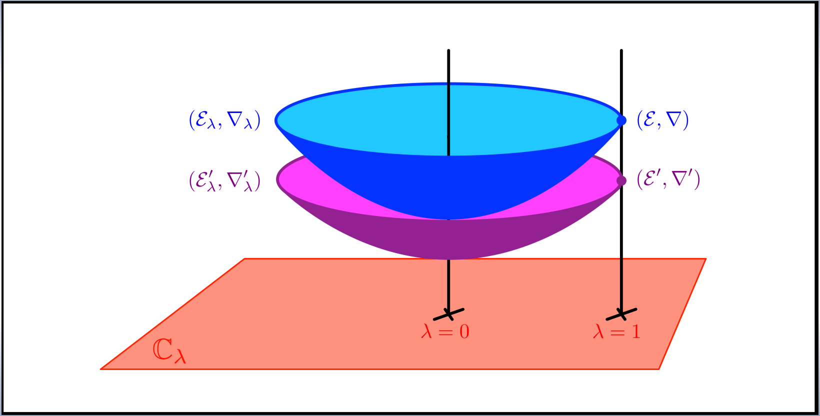
Flatness: $C(\nabla) = 0$.

$\lambda = 1 \rightsquigarrow$ Flat connections
$\lambda = 0 \rightsquigarrow$ Higgs fields

Families of flat λ-connections



Families of flat λ-connections



Higgs bundles and local systems

Theorem (Simpson '92)

There is a fully faithful functor

$$\left\{ (\mathcal{E}, \nabla) \,\middle|\, \begin{array}{c} \mathcal{E} : \text{v.b. of rank } n \text{ on } X \\ \nabla : \text{flat connection} \end{array} \right\} \hookrightarrow \left\{ (\mathcal{F}, \phi) \,\middle|\, \begin{array}{c} \mathcal{F} : \text{v.b. of rank } n \text{ on } X \\ \phi : \text{Higgs field} \end{array} \right\}$$

with essential image given by those semistable Higgs bundles with

$$ch_1(\mathcal{F}) \cdot [\omega]^{\dim_{\mathbb{C}} X - 1} = ch_2(\mathcal{F}) \cdot [\omega]^{\dim_{\mathbb{C}} X - 2} = 0$$

A Higgs bundle $(\mathcal{F}, \phi)$ is *semistable* if

$$\forall \mathcal{F}' \subset \mathcal{F} : \phi(\mathcal{F}') \subset \mathcal{F}' \otimes \Omega^1_X \Longrightarrow \mu(\mathcal{F}') \leq \mu(\mathcal{F})$$

Also: G – linear algebraic group / $\mathbb{C}$:
Tannaka duality $\rightsquigarrow$ a version for principal G-bundles.

The descent-type definition

Choose:

- a cover $\mathfrak{U} = \{U_i\}_{i \in I}$ of X.

Proposition

A **vector bundle** on X is a collection

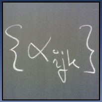
$$\left(\underline{\mathcal{E}} = \{\mathcal{E}_i\}_{i \in I}, \underline{g} = \{g_{ij}\}_{i,j \in I}\right)$$

of vector bundles $\mathcal{E}_i$ on U_i and isomorphisms

$$g_{ij} : \mathcal{E}_j\big|_{U_{ij}} \to \mathcal{E}_i\big|_{U_{ij}}$$

satisfying $g_{ii} = \mathrm{id}_{\mathcal{E}_i}$, $g_{ij} = g_{ji}^{-1}$ and

$$g_{ij} g_{jk} g_{ki} = \mathrm{id}_{\mathcal{E}_i}$$



The descent-type definition

Choose:

- a cover $\mathfrak{U} = \{U_i\}_{i \in I}$ of X, and
- a 2-cocycle $\underline{\alpha} = \{\alpha_{ijk}\} \in \check{Z}^2(\mathfrak{U}, \mathcal{O}_X^\times)$; let $\alpha = [\underline{\alpha}] \in H^2(X, \mathcal{O}_X^\times)$.

Definition

An α-**twisted vector bundle** on X is a collection

$$\left(\underline{\mathcal{E}} = \{\mathcal{E}_i\}_{i \in I}, \underline{g} = \{g_{ij}\}_{i,j \in I}\right)$$

of vector bundles $\mathcal{E}_i$ on U_i and isomorphisms

$$g_{ij} : \mathcal{E}_j\big|_{U_{ij}} \to \mathcal{E}_i\big|_{U_{ij}}$$

satisfying $g_{ii} = \mathrm{id}_{\mathcal{E}_i}$, $g_{ij} = g_{ji}^{-1}$ and

$$g_{ij} g_{jk} g_{ki} = \alpha_{ijk} \, \mathrm{id}_{\mathcal{E}_i}$$

Three remarks

- **Torsion of $\alpha \in H^2(X, \mathcal{O}_X^\times)$:**

$$\left(\underline{\mathcal{E}}, \underline{g}\right) \text{ - } \alpha\text{-twisted vector bundle of rank } n$$
$$\Downarrow$$
$$\det(g_{ij})\det(g_{jk})\det(g_{ki}) = \alpha_{ijk}^n$$
$$\Updownarrow$$
$$\det \underline{g} \in \check{C}^1(\mathfrak{U}, \mathcal{O}_X^\times) \text{ and } \delta_{\check{\text{C}}\text{ech}} \det \underline{g} = \underline{\alpha}^n$$

- **Projectivizing a twisted vector bundle:**

$$\left(\underline{\mathcal{E}}, \underline{g}\right) \text{ - } \alpha\text{-twisted vector bundle of rank } n$$
$$\Downarrow$$
$$\left(\mathbb{P}\underline{\mathcal{E}}, \mathbb{P}\underline{g}\right) \text{ - honest } \mathbb{P}^{n-1}\text{-bundle}$$

- **Lifting $\mathbb{P}^{n-1}$-bundles:**

$$H^1\left(X, GL_n(\mathcal{O}_X)\right) \to H^1\left(X, \mathbb{P}GL_n(\mathcal{O}_X)\right) \to H^2\left(X, \mathcal{O}_X^\times\right)$$

Adding connections

Choose:
- $\underline{\omega} = \{\omega_{ij}\} \in \check{C}^1(\mathfrak{U}, \Omega^1_X)$, and
- $\underline{F} = \{F_i\} \in \check{C}^0(\mathfrak{U}, \Omega^2_X)$.

Let:
- $(\underline{\mathcal{E}}, \underline{g})$ – α-twisted vector bundle, and
- $\underline{\nabla} = \{\nabla_i\}_{i \in I}$, where ∇_i is a connection on $\mathcal{E}_i$

satisfy:
- $C(\nabla_i) = F_i \in \Gamma(U_i, \mathscr{E}nd_{\mathcal{O}_X}(\mathcal{E}_i) \otimes \Omega^2_X)$, and
- $\nabla_i - g_{ij} \nabla_j g_{ij}^{-1} = \omega_{ij} \in \Gamma(U_{ij}, \mathscr{E}nd_{\mathcal{O}_X}(\mathcal{E}_i) \otimes \Omega^1_X)$.

Compatibility conditions:

$$\begin{aligned} dF_i &= 0 \\ F_i - F_j &= d\omega_{ij} \\ \omega_{ik} &= \omega_{ij} + \omega_{jk} + d\log \alpha_{ijk} \end{aligned} \quad \Longleftrightarrow \quad \begin{gathered} (\underline{\alpha}, \underline{\omega}, \underline{F}) \in \check{\mathbb{Z}}^2(\mathfrak{U}, \mathrm{dR}^\times_X) \\ \mathrm{dR}^\times_X := \left[\mathcal{O}^\times_X \xrightarrow{d\log} \Omega^1_X \xrightarrow{d} \Omega^2_X \xrightarrow{d} \cdots \right] \end{gathered}$$

Adding Higgs fields

Choose:

- $\underline{\omega'} = \{\omega'_{ij}\} \in \check{C}^1(\mathfrak{U}, \Omega^1_X)$, and
- $\underline{F'} = \{F'_i\} \in \check{C}^0(\mathfrak{U}, \Omega^2_X)$.

Let:

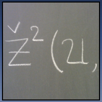
- $\left(\underline{\mathcal{F}}, \underline{g'}\right)$ – α'–twisted vector bundle, and
- $\underline{\phi} = \{\phi_i\}_{i \in I}$, where ϕ_i is a (fat) Higgs field on $\mathcal{F}_i$

satisfy:

- $C(\phi_i) = F'_i \in \Gamma(U_i, \mathscr{E}nd_{\mathcal{O}_X}(\mathcal{F}_i) \otimes \Omega^2_X)$, and
- $\phi_i - g'_{ij}\phi_j(g'_{ij})^{-1} = \omega'_{ij} \in \Gamma(U_{ij}, \mathscr{E}nd_{\mathcal{O}_X}(\mathcal{F}_i) \otimes \Omega^1_X)$.

Compatibility conditions:

$$\begin{array}{c} F'_i - F'_j = 0 \\ \omega'_{ik} = \omega'_{ij} + \omega'_{jk} \end{array} \iff \begin{array}{c} \left(\underline{\alpha'}, \underline{\omega'}, \underline{F'}\right) \in \check{\mathbb{Z}}^2(\mathfrak{U}, \mathrm{Dol}^\times_X) \\ \mathrm{Dol}^\times_X := \left[\mathcal{O}^\times_X \xrightarrow{0} \Omega^1_X \xrightarrow{0} \Omega^2_X \xrightarrow{0} \cdots\right] \end{array}$$



A little bookkeeping...

Definition

$\underline{\alpha} \in \check{Z}^2(\mathfrak{U}, \mathcal{O}_X^\times)$	$\mathfrak{U}$-$\mathbb{G}_m$-gerbe over X
$(\underline{\mathcal{E}}, \underline{g})$	***Basic*** **vector bundle on** $\underline{\alpha}$
$(\underline{\alpha}, \underline{\omega}, \underline{F}) \in \check{\mathbb{Z}}^2(\mathfrak{U}, \mathrm{dR}_X^\times)$	$\mathfrak{U}$-$\mathbb{G}_m$-gerbe with flat connection over X
$(\underline{\mathcal{E}}, \underline{g}, \underline{\nabla})$	***Basic*** **vector bundle on** $(\underline{\alpha}, \underline{\omega}, \underline{F})$
$(\underline{\alpha'}, \underline{\omega'}, \underline{F'}) \in \check{\mathbb{Z}}^2(\mathfrak{U}, \mathrm{Dol}_X^\times)$	Higgs $\mathfrak{U}$-$\mathbb{G}_m$-gerbe over X
$(\underline{\mathcal{F}}, \underline{g'}, \underline{\phi})$	***Basic*** **vector bundle on** $(\underline{\alpha'}, \underline{\omega'}, \underline{F'})$

Main theorem (the cocycle picture)

Theorem (G-R)

Given a $\mathfrak{U}$-$\mathbb{G}_m$-gerbe with flat connection over X, $(\underline{\alpha}, \underline{\omega}, \underline{F})$, there is a Higgs $\mathfrak{U}$-$\mathbb{G}_m$-gerbe over X, $(\underline{\alpha'}, \underline{\omega'}, \underline{F'})$, such that we have a fully faithful functor

$$\left\{ \begin{array}{c} \text{Basic v.b. } (\underline{\mathcal{E}}, \underline{g}, \underline{\nabla}) \\ \text{of rank } n \text{ on } (\underline{\alpha}, \underline{\omega}, \underline{F}) \end{array} \right\} \hookrightarrow \left\{ \begin{array}{c} \text{Basic v.b. } (\underline{\mathcal{F}}, \underline{g'}, \underline{\phi}) \\ \text{of rank } n \text{ on } (\underline{\alpha'}, \underline{\omega'}, \underline{F'}) \end{array} \right\}$$

Conversely, given a Higgs $\mathfrak{U}$-$\mathbb{G}_m$-gerbe over X, $(\underline{\alpha'}, \underline{\omega'}, \underline{F'})$, there is a $\mathfrak{U}$-$\mathbb{G}_m$-gerbe with flat connection over X, $(\underline{\alpha}, \underline{\omega}, \underline{F})$, such that the same conclusion holds.

Why reformulate?

Four reasons:

- Is this statement independent of choices?
- What is the essential image?
- Why is it natural to allow for central twistings?
- Not a particularly beautiful statement...

Principal G-bundles

G – linear algebraic group / $\mathbb{C}$ (e.g., GL_n, $\mathbb{P}GL_n$, SL_n).

- **Definition 1:** locally a cartesian product
  - A fiber bundle $P \to \mathfrak{X}$ of complex-analytic manifolds, with fiber G
  - equipped with a reduction of structure group to $G \subset \operatorname{Aut} G$.
- **Definition 2:** total space with an action of the group
  - A complex-analytic manifold P
  - equipped with a free, proper, holomorphic right G-action ($\Rightarrow \mathfrak{X} := P/G$ is a complex-analytic manifold).
- **Definition 3:** morphism from the base to the classifying stack
  - A complex-analytic manifold $\mathfrak{X}$
  - equipped with a holomorphic map $\mathfrak{X} \to BG$.

$$
\begin{tikzcd}
P \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & EG \simeq * \arrow[d] \\
\mathfrak{X} \arrow[r, "{[P]}"'] & BG
\end{tikzcd}
$$

Principal G-bundles

G – linear algebraic group / $\mathbb{C}$ (e.g., GL_n, $\mathbb{P}GL_n$, SL_n).

- **Definition 2′:** total space with an action of the group
  - A complex-analytic manifold P
  - equipped with a holomorphic right G-action
    ($\Rightarrow \mathfrak{X} := [P/G]$ is a complex-analytic 1-stack).
- **Definition 2″:** total space with an action of the group
  - A complex-analytic ∞-stack P
  - equipped with a holomorphic right G-action.
- **Definition 3″:** morphism to the classifying stack
  - A complex-analytic ∞-stack P
  - equipped with a holomorphic map $\mathfrak{X} \to BG$.

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & EG \simeq * \arrow[d] \\ \mathfrak{X} \arrow[r, "{[P]}"'] & BG \end{tikzcd}$$

Principal G-bundles

G – linear algebraic group / $\mathbb{C}$ (e.g., GL_n, $\mathbb{P}GL_n$, SL_n).

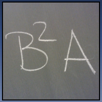
Definition (Principal G-bundle, or G-torsor)

$$\mathrm{Bun}_G := G - \underline{\mathrm{St}} \simeq \underline{\mathrm{St}}_{/BG}$$

- Objects: $\pi_0 \mathrm{Bun}_G(\mathfrak{X}) \cong H^1(\mathfrak{X}, G)$.
- Morphisms:

$$\begin{tikzcd}
P \arrow[rr, "G\text{-equiv.}"] \arrow[d] & & Q \arrow[d] \\
\mathfrak{X} \arrow[rr] \arrow[dr, "{[P]}"'] & & \mathfrak{Y} \arrow[dl, "{[Q]}"] \\
& BG &
\end{tikzcd}$$

$$\begin{aligned}
\mathrm{Map}_G(P, Q) &\simeq \mathrm{Map}_{/BG}(\mathfrak{X}, \mathfrak{Y}) \\
&\simeq \mathrm{Map}(\mathfrak{X}, \mathfrak{Y}) \times^h_{\mathrm{Map}(\mathfrak{X}, BG), [P]} *
\end{aligned}$$



Principal BA-bundles

A – *abelian* linear algebraic group / $\mathbb{C}$ (e.g., $\mathbb{G}_m$, $\mathbb{G}_a$, μ_n)

$\Rightarrow$ BA – abelian group stack / $\mathbb{C}$.

Definition (Principal BA-bundle, or A-gerbe)

$$\mathrm{Bun}_{BA} := BA - \underline{\mathrm{St}} \simeq \underline{\mathrm{St}}_{/B^2A}$$

- Objects: $\pi_0 \mathrm{Bun}_{BA}(\mathfrak{X}) \cong H^2(\mathfrak{X}, A)$.
- Morphisms:

$$\begin{tikzcd}
{}_\beta\mathfrak{X} \arrow[rr, "BA\text{-equiv.}"] \arrow[d] & & {}_\gamma\mathfrak{Y} \arrow[d] \\
\mathfrak{X} \arrow[rr] \arrow[dr, "\beta"'] & & \mathfrak{Y} \arrow[dl, "\gamma"] \\
& B^2A &
\end{tikzcd}$$

$$\mathrm{Map}_{BA}({}_\beta\mathfrak{X}, {}_\gamma\mathfrak{Y}) \simeq \mathrm{Map}_{/B^2A}(\mathfrak{X}, \mathfrak{Y})$$
$$\simeq \mathrm{Map}(\mathfrak{X}, \mathfrak{Y}) \times^h_{\mathrm{Map}(\mathfrak{X}, B^2A), \beta} *$$

Nikolaus, Schreiber, Stevenson, Principal ∞-bundles - General theory, `arXiv:1207.0248`

Vector bundles on $\mathbb{G}_m$-gerbes

Two $\mathbb{G}_m$-gerbes:

$$\alpha \in H^2(X, \mathcal{O}_X^\times) \qquad\qquad 1 \to \mathbb{G}_m \to GL_n \to \mathbb{P}GL_n \to 1$$

$$\begin{tikzcd} {}_\alpha X \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & * \arrow[d] \\ X \arrow[r, "\alpha"'] & B^2\mathbb{G}_m \end{tikzcd} \qquad\qquad \begin{tikzcd} BGL_n \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & * \arrow[d] \\ B\mathbb{P}GL_n \arrow[r] & B^2\mathbb{G}_m \end{tikzcd}$$

Proposition

$$\left\{ \begin{array}{c} \text{Basic vector bundles} \\ \text{of rank } n \text{ on } {}_\alpha X \end{array} \right\} \simeq \mathrm{Map}_{B\mathbb{G}_m}({}_\alpha X, BGL_n)$$

$$\simeq \mathrm{Map}(X, B\mathbb{P}GL_n) \times^h_{\mathrm{Map}(X, B^2\mathbb{G}_m), \alpha} *$$

The de Rham stack

Definition

$$X_{\mathrm{dR}} := \left[(X \times X)^{\wedge}_{\Delta} \rightrightarrows X\right]$$

The **de Rham stack** of X codifies connections:

- $\mathrm{Map}(X_{\mathrm{dR}}, BG) \simeq \left\{ \begin{array}{c} \text{ppal. } G\text{-bundles on } X \\ \text{with flat connection} \end{array} \right\}$
- $\mathrm{Map}(X_{\mathrm{dR}}, B^2\mathbb{G}_m) \simeq \left\{ \begin{array}{c} \mathbb{G}_m\text{-gerbes over } X \\ \text{with flat connection} \end{array} \right\} \leftrightsquigarrow \mathbb{H}^2(X, \mathrm{dR}^{\times}_X)$

Proposition

$$\left\{ \begin{array}{c} \text{Basic vector bundles} \\ \text{of rank } n \text{ on } {}_{\theta}(X_{\mathrm{dR}}) \end{array} \right\} \simeq \mathrm{Map}_{B\mathbb{G}_m}({}_{\theta}(X_{\mathrm{dR}}), BGL_n)$$

$$\simeq \mathrm{Map}(X_{\mathrm{dR}}, B\mathbb{P}GL_n) \times^{h}_{\mathrm{Map}(X_{\mathrm{dR}}, B^2\mathbb{G}_m), \theta} *$$

The Dolbeault stack

Definition

$$X_{\mathrm{Dol}} := \left[(TX)_0^{\wedge} \rightrightarrows X \right]$$

The **Dolbeault stack** of X codifies Higgs fields:

- $\mathrm{Map}(X_{\mathrm{Dol}}, BG) \simeq \left\{ \begin{array}{c} \text{ppal. } G\text{-bundles on } X \\ \text{with Higgs field} \end{array} \right\}$
- $\mathrm{Map}(X_{\mathrm{Dol}}, B^2\mathbb{G}_m) \simeq \{\text{Higgs } \mathbb{G}_m\text{-gerbes over } X\} \leftrightsquigarrow \mathbb{H}^2(X, \mathrm{Dol}_X^{\times})$

Proposition

$$\left\{ \begin{array}{l} \text{Basic vector bundles} \\ \text{of rank } n \text{ on } {}_{\theta'}(X_{\mathrm{Dol}}) \end{array} \right\} \simeq \mathrm{Map}_{B\mathbb{G}_m}\left({}_{\theta'}(X_{\mathrm{Dol}}), BGL_n\right)$$

$$\simeq \mathrm{Map}(X_{\mathrm{Dol}}, B\mathbb{P}GL_n) \times^h_{\mathrm{Map}(X_{\mathrm{Dol}}, B^2\mathbb{G}_m), \theta'} *$$

Main theorem

Theorem (G-R)

Given $\theta \in \mathrm{Map}(X_{\mathrm{dR}}, B^2\mathbb{G}_m)$ *there exists* $\theta' \in \mathrm{Map}(X_{\mathrm{Dol}}, B^2\mathbb{G}_m)$ *such that we have a fully faithful functor*

$$\mathrm{Map}_{B\mathbb{G}_m}\big({}_{\theta}(X_{\mathrm{dR}}), BGL_n\big) \hookrightarrow \mathrm{Map}_{B\mathbb{G}_m}\big({}_{\theta'}(X_{\mathrm{Dol}}), BGL_n\big)$$

Given $\theta' \in \mathrm{Map}(X_{\mathrm{Dol}}, B^2\mathbb{G}_m)$ *there is* $\theta \in \mathrm{Map}(X_{\mathrm{dR}}, B^2\mathbb{G}_m)$ *such that the same conclusion holds.*

The obvious approach

$$
\begin{tikzcd}
\mathrm{Map}_{B\mathbb{G}_m}(\theta(X_{\mathrm{dR}}), BGL_n) \arrow[r] \arrow[d] & * \arrow[d, "\theta"] \\
\mathrm{Map}(X_{\mathrm{dR}}, B\mathbb{P}GL_n) \arrow[r] \arrow[d, "\mathrm{nah}"', "\simeq"] & \mathrm{Map}(X_{\mathrm{dR}}, B^2\mathbb{G}_m) \arrow[d, dashed, "\simeq"', "?"] \\
\mathrm{Map}(X_{\mathrm{Dol}}, B\mathbb{P}GL_n)^{\mathrm{ss},0} \arrow[r] & \mathrm{Map}(X_{\mathrm{Dol}}, B^2\mathbb{G}_m) \\
\mathrm{Map}_{B\mathbb{G}_m}(\theta'(X_{\mathrm{Dol}}), BGL_n)^{\mathrm{pss}} \arrow[r] \arrow[u] & * \arrow[u, "\theta'"']
\end{tikzcd}
$$

Hodge correspondence for gerbes

Proposition (G-R)

If $A \simeq \mathbb{G}_a^{\oplus m} \oplus F$, we have

$$\operatorname{Map}\left(X_{\mathrm{dR}}, B^2 A\right) \simeq \operatorname{Map}\left(X_{\mathrm{Dol}}, B^2 A\right)$$

- For $A = \mathbb{G}_a$, classical abelian Hodge theory.
- For $A = F$, the connection and Higgs data are trivial ($\mathfrak{f} = 0$).
- For $A = \mathbb{G}_m$:

$$\begin{array}{ccc} \Omega \operatorname{Map}\left(X_{\mathrm{dR}}, B^2\mathbb{G}_m\right) & \simeq & \operatorname{Map}\left(X_{\mathrm{dR}}, B\mathbb{G}_m\right) \\ & & \not\simeq \\ \Omega \operatorname{Map}\left(X_{\mathrm{Dol}}, B^2\mathbb{G}_m\right) & \simeq & \operatorname{Map}\left(X_{\mathrm{Dol}}, B\mathbb{G}_m\right) \end{array}$$

Torsion to the rescue

$$
\begin{tikzcd}
\mathrm{Map}_{B\mathbb{G}_m}({}_{\theta}(X_{\mathrm{dR}}), BGL_n) \arrow[rr] \arrow[d] & & * \arrow[d, "\theta"] \arrow[dl, bend right, "\widetilde{\theta}"'] \\
\mathrm{Map}(X_{\mathrm{dR}}, B\mathbb{P}GL_n) \arrow[rr] \arrow[dr, bend right] \arrow[ddd, "\mathrm{nah}"', "\simeq"] & & \mathrm{Map}(X_{\mathrm{dR}}, B^2\mathbb{G}_m) \\
 & \mathrm{Map}(X_{\mathrm{dR}}, B^2\mu_n) \arrow[ur, bend right] \arrow[d, "\simeq"'] & \\
 & \mathrm{Map}(X_{\mathrm{Dol}}, B^2\mu_n) \arrow[dr, bend left] & \\
\mathrm{Map}(X_{\mathrm{Dol}}, B\mathbb{P}GL_n)^{\mathrm{ss},0} \arrow[rr] \arrow[ur, bend left] & & \mathrm{Map}(X_{\mathrm{Dol}}, B^2\mathbb{G}_m) \\
\mathrm{Map}_{B\mathbb{G}_m}({}_{\theta'}(X_{\mathrm{Dol}}), BGL_n)^{\mathrm{pss}} \arrow[rr] \arrow[u] & & * \arrow[u, "\theta'"'] \arrow[uul, bend left, "\widetilde{\theta}'"]
\end{tikzcd}
$$

Some homotopy theory. . .

$$
\begin{tikzcd}
\mathrm{Map}_{B\mathbb{G}_m}(\theta(X_{\mathrm{dR}}), BGL_n) \arrow[rr] \arrow[d] & & * \arrow[d, "\theta"] \arrow[dl, bend right, "\widetilde{\theta}"'] \\
\mathrm{Map}(X_{\mathrm{dR}}, B\mathbb{P}GL_n) \arrow[rr] \arrow[dr, bend right] & & \mathrm{Map}(X_{\mathrm{dR}}, B^2\mathbb{G}_m) \\
& \mathrm{Map}(X_{\mathrm{dR}}, B^2\mu_n) \arrow[ur, bend right] &
\end{tikzcd}
$$

$$
\mathrm{Map}_{B\mathbb{G}_m}(\theta(X_{\mathrm{dR}}), BGL_n) \simeq
$$

$$
\coprod_{\text{liftings}} \frac{\mathrm{Map}_{B\mu_n}(\widetilde{\theta}(X_{\mathrm{dR}}), BSL_n) \times \mathrm{Map}(X_{\mathrm{dR}}, B\mathbb{G}_m)}{\mathrm{Map}(X_{\mathrm{dR}}, B\mu_n)}
$$

Some homotopy theory...

$$\mathrm{Map}_{B\mathbb{G}_m}\big(\theta'(X_{\mathrm{Dol}}), BGL_n\big)^{\mathrm{pss}} \simeq$$

$$\coprod_{\text{liftings}} \frac{\mathrm{Map}_{B\mu_n}\big(\widetilde{\theta'}(X_{\mathrm{Dol}}), BSL_n\big) \times \mathrm{Map}(X_{\mathrm{Dol}}, B\mathbb{G}_m)}{\mathrm{Map}(X_{\mathrm{Dol}}, B\mu_n)}$$

$$\begin{tikzcd}
 & \mathrm{Map}(X_{\mathrm{Dol}}, B^2\mu_n) \arrow[dr, bend left] & \\
\mathrm{Map}(X_{\mathrm{Dol}}, B\mathbb{P}GL_n)^{\mathrm{ss},0} \arrow[ur, bend left] \arrow[rr] & & \mathrm{Map}(X_{\mathrm{Dol}}, B^2\mathbb{G}_m) \\
\mathrm{Map}_{B\mathbb{G}_m}\big(\theta'(X_{\mathrm{Dol}}), BGL_n\big)^{\mathrm{pss}} \arrow[u] \arrow[rr] & & * \arrow[u, "\theta'"'] \arrow[uul, bend left, "\widetilde{\theta'}"]
\end{tikzcd}$$

The last condition becomes evident

$$\mathrm{Map}_{B\mathbb{G}_m}\big(\theta'(X_{\mathrm{Dol}}), BGL_n\big)^{ss} \simeq$$

$$\coprod_{\text{liftings}} \frac{\mathrm{Map}_{B\mu_n}\big(\widetilde{\theta'}(X_{\mathrm{Dol}}), BSL_n\big) \times \mathrm{Map}(X_{\mathrm{Dol}}, B\mathbb{G}_m)^0}{\mathrm{Map}(X_{\mathrm{Dol}}, B\mu_n)}$$

$$\text{tnah} \;\; \big| \!\wr$$

$$\mathrm{Map}_{B\mathbb{G}_m}\big(\theta(X_{\mathrm{dR}}), BGL_n\big) \simeq$$

$$\coprod_{\text{liftings}} \frac{\mathrm{Map}_{B\mu_n}\big(\widetilde{\theta}(X_{\mathrm{dR}}), BSL_n\big) \times \mathrm{Map}(X_{\mathrm{dR}}, B\mathbb{G}_m)}{\mathrm{Map}(X_{\mathrm{dR}}, B\mu_n)}$$

Generalization to other groups

The master diagram for GL_n:

$$\begin{tikzcd}
1 \arrow[r] & \mu_n \arrow[r] \arrow[d, hook] & SL_n \arrow[r] \arrow[d, hook] & \mathbb{P}GL_n \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathbb{G}_m \arrow[r] \arrow[d, two heads, "(-)^n"] & GL_n \arrow[r] \arrow[d, two heads, "\det"] & \mathbb{P}GL_n \arrow[r] & 1 \\
& \mathbb{G}_m \arrow[r, equal] & \mathbb{G}_m & &
\end{tikzcd}$$

- GL_n-bundles twisted by $\alpha \in H^2(\mathfrak{X}, \mathbb{G}_m)$.
- $\alpha \in \operatorname{im}\left\{H^2(\mathfrak{X}, \mu_n) \to H^2(\mathfrak{X}, \mathbb{G}_m)\right\}$.
- They induce untwisted $\mathbb{P}GL_n$-bundles.
- SL_n-bundles twisted by lifts of α to $H^2(\mathfrak{X}, \mu_n)$ in the proof.

Generalization to other groups

The **master diagram** for GL_n:

$$\begin{tikzcd}
1 \arrow[r] & \mu_n \arrow[r] \arrow[d, hook] & SL_n \arrow[r] \arrow[d, hook] & \mathbb{P}GL_n \arrow[r] \arrow[d, equal] & 1 \\
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 & \mathbb{G}_m \arrow[r, equal] & \mathbb{G}_m & &
\end{tikzcd}$$

- $\mathbb{G}_m \to GL_n$ with $\mathbb{G}_m \subseteq Z(GL_n)$.
- $GL_n \xrightarrow{\det} \mathbb{G}_m$
  - det is surjective,
  - $\det|_{\mathbb{G}_m}$ is surjective, and
  - $\ker(\det|_{\mathbb{G}_m})$ contains no tori.

Generalization to other groups

The **master diagram** for H (*connected* linear algebraic group / $\mathbb{C}$):

$$\begin{tikzcd}
1 \arrow[r] & A' \arrow[r] \arrow[d, hook] & H' \arrow[r] \arrow[d, hook] & K \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & A \arrow[r] \arrow[d, two heads, "\kappa|_A"] & H \arrow[r] \arrow[d, two heads, "\kappa"] & K \arrow[r] & 1 \\
 & \mathbb{G}_m^{\oplus r} \arrow[r, equal] & \mathbb{G}_m^{\oplus r} & &
\end{tikzcd}$$

- $A \to H$ with $A \subseteq Z(H)$.
- $H \xrightarrow{\kappa} \mathbb{G}_m^{\oplus r}$
  - κ is surjective,
  - $\kappa|_A$ is surjective, and
  - $\ker(\kappa|_A)$ contains no tori.

Generalization to other groups

The **master diagram** for H (*connected* linear algebraic group / $\mathbb{C}$):

$$\begin{tikzcd}
1 \arrow[r] & A' \arrow[r] \arrow[d, hook] & H' \arrow[r] \arrow[d, hook] & K \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & A \arrow[r] \arrow[d, two heads, "\kappa|_A"] & H \arrow[r] \arrow[d, two heads, "\kappa"] & K \arrow[r] & 1 \\
 & \mathbb{G}_m^{\oplus r} \arrow[r, equal] & \mathbb{G}_m^{\oplus r} & &
\end{tikzcd}$$

- H-bundles twisted by $\alpha \in H^2(\mathfrak{X}, A)$.
- $\alpha \in \operatorname{im}\left\{H^2(\mathfrak{X}, A') \to H^2(\mathfrak{X}, A)\right\}$.
- They induce untwisted K-bundles.
- H'-bundles twisted by lifts of α to $H^2(\mathfrak{X}, A')$ in the proof.