The nonabelian Hodge theorem

Let $X$ be a smooth projective variety over $\mathbb{C}$. The nonabelian Hodge theorem of [Sim02] gives a fully faithful functor of groups

$$
\begin{cases}
\text{Flat vector bundles} \\
\text{of rank } n \text{ on } X
\end{cases} \rightarrow \begin{cases}
\text{Higgs bundles} \\
\text{of rank } n \text{ on } X
\end{cases}
$$

Here a flat vector bundle on $X$ is a pair $(E, \nabla)$ of a vector bundle $E$ together with a flat connection $\nabla$ on it. A Higgs bundle $(F, \phi)$ is a vector bundle $F$ equipped with an $O_X$-linear map $\phi: F \rightarrow F \otimes \Omega^1_X$ with curvature $C(\phi) = -\phi \wedge \phi = 0$.

Twisted vector bundles

Given a cover $U = \{U_i\}$ of $X$ and a Cech 2-cocycle $\alpha = (\alpha_{ijk}) \in Z^2(H^1(X, O_X^*), \mathbb{C})$, an $\alpha$-twisted vector bundle is a collection $\{(E_i), (\gamma_{ij})\}$ of vector bundles $E_i$ on $U_i$ and isomorphisms $\gamma_{ij}: E_i \rightarrow E_j$ satisfying the twisted cocycle condition $\gamma_{ijk} \circ \gamma_{jkl} = (\alpha_{ijk}) \circ \gamma_{ikl}$. Morphisms of $\alpha$-twisted vector bundles are isomorphisms of the locally defined vector bundles that intertwine the transition functions.

Adding connections

Choose cocycles $\{\omega_j\} \in C^0(U, \Omega^1_{X})$ and $\{F_i\} \in C^1(U, \Omega^2_{X})$, and equip each $E_i$ in an $\alpha$-twisted vector bundle with a connection $\nabla_i$ satisfying

$$\nabla_i - g_{ij}^* \nabla_j = \omega_{ij}, \quad C(\nabla_i) = F_i$$

The compatibility conditions make the triple $\{(\omega_j), (\gamma_{ij}), (F_i)\}$ into a 2-cocycle in hypercohomology of the multiplicative de Rham complex

$$d \Omega^1_X = \Omega^1_X \otimes \Omega^2_X \rightarrow \Omega^2_X \rightarrow \Omega^3_X \rightarrow \cdots$$

We call the latter a triple $\alpha$-Gau\-gerbe with flat connection, and say that $\{(E_i), (\nabla_i), (\gamma_{ij})\}$ is a vector bundle on it.

Adding Higgs fields

Choose cocycles $\{\alpha_{ij} \}$ and $\{F_i\}$ such that $\alpha_{ijk} = \gamma_{ijk}^* \alpha_{ij}$. We have

$$\nabla_i - g_{ij}^* \nabla_j = \alpha_{ij}, \quad C(\nabla_i) = F_i$$

The triple $\{(\alpha_{ij}), (\gamma_{ij}), (F_i)\}$ assembles into a 2-cocycle in hypercohomology of the multiplicative Dolbeault complex

$$d \Omega^1_X = \Omega^1_X \otimes \Omega^2_X \rightarrow \Omega^2_X \rightarrow \Omega^3_X \rightarrow \cdots$$

We call this a Higgs $\alpha$-Gau\-gerbe, and the triple $\{(E_i), (\phi_i), (g_{ij})\}$ a vector bundle on it.

Codifying twisted vector bundles: gerbes

$G_{\alpha}$-gerbes [Gir71] over $X$ (with hand in the terminology of loc.cit.) are certain I-stacks locally isomorphic to $BG_{\alpha}$. They provide geometric representatives of classes in $H^1(X, O_X^*)$. As principal $BG_{\alpha}$-bundles [NSS12], we have a classifying $(2)$-stack for them:

$$\alpha X \rightarrow * \xrightarrow{\alpha} * \xrightarrow{\alpha} B G_{\alpha}$$

We can realize $\alpha$-twisted vector bundles on $X$ as those vector bundles on the gerbe $\alpha X$ classified by $\alpha = \{\omega\} \in H^1(X, O_X^*)$ whose classifying morphism $\alpha X \rightarrow B G_{\alpha}$ is $BG_{\alpha}$-equivariant. We have

$$BG_{\alpha}(\alpha X, B G_{\alpha}) \simeq (X, B P G_{\alpha}) \times_{(X, B G_{\alpha})} *$$

Notation: for $X$ and $Y$ objects in an $\infty$-topos, $(X, Y)$ denotes the $\infty$-groupoid of morphisms from $X$ to $Y$. If they come equipped with an action of $G$, we denote by $G(X, Y)$ the $\infty$-groupoid of $G$-equivariant morphisms.

Codifying connections: the de Rham stack

The de Rham stack $X_{dr}$ of $X$ is the quotient of $X$ by the formal neighborhood of the diagonal in $X \times X$. It encodes flat connections:

- vector bundles on it are flat vector bundles on $X$;
- we call $G_{\alpha}$-gerbes over it $G_{\alpha}$-gerbes with flat connection over $X$.

Equivalence classes of the latter are given by $H^1(X, dR_X)$.

Given a $G_{\alpha}$-gerbe over $X$, vector bundles over it (with an equivariance condition) can be interpreted as twisted vector bundles with connection compatible with that of the gerbe:

$$BG_{\alpha}(\alpha X, B G_{\alpha}) \simeq (X_{dr}, B P G_{\alpha}) \times_{(X, B G_{\alpha})} *$$

Codifying Higgs fields: the Dolbeault stack

The Dolbeault stack $X_{dol}$ of $X$ is the quotient of $X$ by the formal neighborhood of the zero section of the tangent bundle of $X$. It encodes Higgs fields:

- vector bundles on it are Higgs bundles on $X$;
- we call $G_{\alpha}$-gerbes over it $G_{\alpha}$-gerbes over $X$. Equivalence classes of the latter are given by $H^1(X, Dol_X)$.

Given a $G_{\alpha}$-gerbe over $X$, vector bundles over it (with an equivariance condition) can be interpreted as twisted vector bundles with a Higgs field compatible with the Higgs data of the gerbe:

$$BG_{\alpha}(\alpha X, B G_{\alpha}), B L_{\alpha} \simeq (X_{dol}, B P G_{\alpha}) \times_{(X, B G_{\alpha})} *$$

Main theorem (the cocycle picture)

Given a $\alpha$-Gau\-gerbe with flat connection $\{(\alpha_{ij}), (\omega_{ij}), (F_i)\}$, there is a Higgs $\alpha$-Gau\-gerbe $\{(\alpha'_{ij}), (\omega'_j), (F'_j)\}$ such that we have a fully faithful functor

$$\begin{cases}
\text{Vector bundles of rank } n \\
\text{on } \alpha X, (\omega_{ij}), (F_i), \phi_i
\end{cases} \rightarrow \begin{cases}
\text{Vector bundles of rank } n \\
\text{on } \{(\alpha'_{ij}), (\omega'_j), (F'_j)\}
\end{cases}$$

Conversely, given a Higgs $\alpha$-Gau\-gerbe $\{(\alpha_{ij}), (\omega_{ij}), (F_i)\}$, there is a $\alpha$-Gau\-gerbe $\{(\alpha'_{ij}), (\omega'_j), (F'_j)\}$ such that the same conclusion holds.

Main theorem (intrinsinc formulation)

Given $\theta \in (X_{dol}, B G_{\alpha})$ there exists $\theta' \in (X_{dol}, B G_{\alpha})$ such that we have a fully faithful functor

$$BG_{\alpha}(\alpha X, B G_{\alpha}) \rightarrow BG_{\alpha}(\theta X_{dol}, B G_{\alpha})$$

Conversely, given $\theta' \in (X_{dol}, B G_{\alpha})$, there is $\theta \in (X_{dol}, B G_{\alpha})$ such that the same conclusion holds.

The proof

The projective bundles on both sides are related by the nonabelian Hodge theorem [Sim02]. The gerbes are torsion (otherwise the two sides are empty and the correspondence is trivial), and results of [GR11] allow us to relate them using the higher nonabelian Hodge correspondence of [Sim02].

References