Poisson’s equation

In the above problems, we have been working with harmonic functions. The reason why Complex Analysis is so helpful with those is because holomorphic functions transform harmonic functions to harmonic functions.

There are other cases in which holomorphic functions might help transform some problems into easier ones. For example, electrostatic potentials in the presence of nonzero charge densities (recall Maxwell’s equations above) satisfy Poisson’s equation

\[ \Delta V = \nabla \cdot (\nabla V) = -\nabla \cdot E = -\frac{\rho}{\epsilon_0} \]

The following two problems explore how functions satisfying a two-dimensional Poisson equation are transformed under composition with a holomorphic function.

5.1 Suppose that a holomorphic function \( w = f(z) = u(x, y) + iv(x, y) \) maps a domain \( D_z \) in the \( z \) plane onto a domain \( D_w \) in the \( w \) plane; and let a function \( p(u, v) \), with continuous partial derivatives of the first and second order, be defined on \( D_w \). Use the chain rule for partial derivatives to show that if \( P(x, y) = p(u(x, y), v(x, y)) \), then

\[ P_{xx}(x, y) + P_{yy}(x, y) = \left[ p_{uu}(u, v) + p_{vv}(u, v) \right] |f'(z)|^2 \]

**Hint:** in the simplifications you will need to use the Cauchy–Riemann equations for \( f \).

By the chain rule and the product rule,

\[ P_x = p_u u_x + p_v v_x, \]

\[ P_{xx} = p_{uu}(u_x)^2 + p_{uv} u_x v_x + p_{u} u_{xx} + p_{vv}(v_x)^2 + p_{vu} u_x v_x + p_{v} v_{xx}. \]

Similarly,

\[ P_y = p_u u_y + p_v v_y, \]

\[ P_{yy} = p_{uu}(u_y)^2 + p_{uv} u_y v_y + p_{u} u_{yy} + p_{vv}(v_y)^2 + p_{vu} u_y v_y + p_{v} v_{yy}. \]

Adding these together and collecting like terms, we see that

\[ P_{xx} + P_{yy} = p_{uu} \left[(u_x)^2 + (u_y)^2\right] + p_{vv} \left[(v_x)^2 + (v_y)^2\right] \]

\[ + (p_{uv} + p_{vu})(u_x v_x + u_y v_y) + p_u(u_{xx} + u_{yy}) + p_v(v_{xx} + v_{yy}). \]

Since \( u \) and \( v \) are the real and imaginary parts of a function, \( f \), that is holomorphic on \( D_z \), they satisfy the Cauchy-Riemann equations on said domain. It follows that

\[ u_x v_x + u_y v_y = -u_x u_y + u_y u_x = 0, \]

\[ * \]
which says that the third term in (∗) vanishes. The harmonicity of \( u \) and \( v \) makes the fourth and fifth terms zero too. One more application of the Cauchy-Riemann equations yields

\[
(u_x)^2 + (u_y)^2 = (u_x)^2 + (-v_x)^2 = |f'(z)|^2, \\
(v_x)^2 + (v_y)^2 = (v_x)^2 + (u_x)^2 = |f'(z)|^2,
\]

making (∗) into the equation in the statement.

5.2 Let \( p(u, v) \) be a function that has continuous partial derivatives of the first and second orders and satisfies Poisson’s equation

\[
p_{uu}(u, v) + p_{vv}(u, v) = \Phi(u, v)
\]

in a domain \( D_w \) of the \( w \) plane, where \( \Phi \) is a prescribed function. Show how it follows from the previous problem that if a holomorphic function \( w = f(z) = u(x, y) + iv(x, y) \) maps a domain \( D_z \) onto the domain \( D_w \), then the function \( P(x, y) = p(u(x, y), v(x, y)) \) satisfies the Poisson equation

\[
P_{xx}(x, y) + P_{yy}(x, y) = \Phi(u(x, y), v(x, y)) |f'(z)|^2
\]

on \( D_z \).

Simply substitute

\[
p_{uu}(u, v) + p_{vv}(u, v) = \Phi(u, v)
\]

into the equation satisfied by \( P \) in the problem above,

\[
P_{xx}(x, y) + P_{yy}(x, y) = \left[ p_{uu}(u, v) + p_{vv}(u, v) \right] |f'(z)|^2
\]

Conformal transformations

Suppose \( w = f(z) \) is conformal at \( z_0 \)—that is, it is holomorphic at \( z_0 \) and \( f'(z_0) \neq 0 \). Recall the setup. We take a path \( C : [a, b] \to \mathbb{C} \) in the \( z \)-plane that passes through \( z_0 \) at time \( t = 0 \) (i.e., \( C(0) = z_0 \)). The image of this path in the \( w \)-plane is \( \Gamma = f \circ C : [a, b] \to \mathbb{C} \); it passes through \( w_0 = f(z_0) \) at time \( t = 0 \). Then the tangent vectors to \( C \) and \( \Gamma \) at the points \( z_0 \) and \( w_0 \), respectively, are related by the equation

\[
\Gamma'(0) = f'(z_0) C'(0).
\]

This is a rotation by an angle \( \arg f'(z_0) \)—the angle of rotation of \( f \) at \( z_0 \)—and a dilation by a factor \( |f'(z_0)| \)—the scale factor of \( f \) at \( z_0 \).

5.3 Determine the angle of rotation at the point \( z = 2 + i \) of the transformation \( f(z) = z^2 \) and illustrate it for the curves \( x = 2 \) and \( y = 1 \) (both of them passing through the point \( 2 + i \), and making a right angle with each other there). Show that the scale factor of \( f \) at that point is \( 2\sqrt{5} \).
We have $f'(2 + i) = 2(2 + i)$, so

$$\arg f'(2 + i) = \arctan(1/2) \cong 0.464, \quad |f'(2 + i)| = \sqrt{4^2 + 2^2} = 2\sqrt{5}.$$ 

Consider the lines $x = 2$ and $y = 1$ that pass through the point $z = 2 + i$. The image of the first one is given by

$$u = 4 - y^2, v = 4y \quad \implies \quad u = 4 - \frac{v^2}{16}$$

The second one is described by

$$u = x^2 - 1, v = 2x \quad \implies \quad u = \frac{v^2}{4} - 1$$

These are two parabolas, one opening in the positive $u$-direction, the other one in the negative $u$-direction. They cross perpendicularly at $(2 + i)^2 = 3 + 4i$.

5.4 Show that the angle of rotation at a nonzero point $z = r_0 e^{i\theta_0}$ under the transformation $f(z) = z^n$ ($n \geq 1$) is $(n - 1)\theta_0$. Determine the scale factor of $f$ at that point.

A straightforward computation yields

$$\arg f'(z_0) = \arg(n z_0^{n-1}) = \arg(nr_0^{n-1} e^{i(n-1)\theta}) = (n - 1)\theta, \quad |f'(z_0)| = \left|nr_0^{n-1} e^{i(n-1)\theta}\right| = nr_0^{n-1}.$$ 

5.5 Fix some $x_0 \in \mathbb{R}$. Consider the Möbius transformation

$$w = T(z) = \frac{z - i}{z + i}$$
It takes the real line to the circle of radius 1 (see Problem 2.6), and \( \mathbb{H} \) to \( \mathbb{D} \). Let \( L_{x_0} \) be the half-line \( x = x_0, y > 0 \). Since it is contained in \( \mathbb{H} \), its image is contained in \( \mathbb{D} \). And because \( L_{x_0} \) is contained in a line in the \( z \)-plane, its image \( T(L_{x_0}) \) is contained in a line or a circle in the \( w \)-plane. We do know two points in (the closure of) \( T(L_{x_0}) \): \((x_0 - i)/(x_0 + i)\) and 1 (the images of \( x_0 \) and \( \infty \), respectively). Explain why the fact that \( L_{x_0} \) is perpendicular to the real axis in the \( z \)-plane is enough to determine \( T(L_{x_0}) \), without having to compute the image of some third point in \( L_{x_0} \).

**Hint:** think of all the possible lines and circles passing through \((x_0 - i)/(x_0 + i)\) and 1, and observe that only one of those is perpendicular to the unit circle at \((x_0 - i)/(x_0 + i)\).

The transformation \( T \) is holomorphic at \( x_0 \), and its derivative does not vanish at that point:

\[
T'(x_0) = \frac{2i}{(z + i)^2} \bigg|_{z=x_0} = \frac{2i}{(x_0 + i)^2} \neq 0
\]

Hence \( T \) is conformal at \( x_0 \). The real line and the line \( x = x_0 \) cross perpendicularly at \( x_0 \). Hence, their images also intersect perpendicularly at \( T(x_0) \). The image of \( x = x_0 \) is a line or a circle passing through the points \( T(x_0) \) and 1, and \( T(L_{x_0}) \) is the portion of the latter that is contained in \( \mathbb{D} \). But there is only one circle passing through \( T(x_0) \) and 1 that is perpendicular to the unit circle at \( T(x_0) \) (and it also happens to be perpendicular to the unit circle at 1!).

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**Critical points**

5.6 Find the critical points of the following functions, as well as their order

(i) \( f(z) = z - i \),  
(ii) \( f(z) = 1 - z^2 \),  
(iii) \( f(z) = z^4 - 2z^2 \).

(i) The derivative \( f'(z) = 1 \) never vanishes, so \( f(z) \) has no critical points.

(ii) Since \( f'(z) = -2z \), the function \( f(z) \) has a unique critical point, located at the origin; it is of order 1, since \( f''(0) = -2 \neq 0 \).
The derivative $f'(z) = 4z^3 - 4z = 4z(z^2 - 1)$ has zeroes at $z = 0, \pm i$. The second derivative $f''(z) = 12z^2 - 4$ is nonzero at each of them, so they are critical points of order 1.

5.7 In this problem, we will analyze an example of a critical point of order 1, and how a holomorphic function looks around it.

Consider the function

$$w = f(z) = \frac{1 - iz^2}{1 - z^3}$$

You could check (but the calculations are a bit gruesome, so either believe me or use a computer algebra system—you could be masochistic and do them yourself, but there’s no point in doing that) that it has a critical point of order 1 at $z = 0$, that $f(0) = 1$ (okay, this one is easy), and that $f''(0) = -2i$. Its Taylor series (we will talk more about these in the future) starts with the following few terms:

$$f(z) = 1 - iz^2 + \cdots$$

In other words, in an $\epsilon$-neighborhood of the origin, it behaves in the same way as the function $g(z) = 1 - iz^2$.

We will try to picture $g(z)$ by breaking it up as a composition of the following transformations:

$$Z_1 = z^2, \quad Z_2 = -iz_1, \quad w = 1 + Z_2$$

These are, respectively, the squaring map, a rotation and a translation—all functions we have discussed in class and you know. Notice that $z = 0$ maps to $Z_1 = 0$, then to $Z_2 = 0$, and finally to $w = 1$. Work your way backwards from the $w$-plane to see what the inverse images of the lines $u = 1$ and $v = 0$ are in the $z$-plane under these series of transformations. That is, draw the appropriate blue and red lines in the $Z_1$- and $z$-planes in Figure 1 (I’ve given you already what the inverse images of $u = 1$ and $v = 0$ are in the $Z_2$-plane). Fill in the colors for the corresponding regions too.

The last map, $Z_2 \mapsto w = 1 + Z_2$ is a translation by 1; its inverse, then, is a translation by $-1$. The map $Z_1 \mapsto Z_2 = iZ_1$ is a rotation by $-\pi/2$; its inverse is a rotation by $\pi/2$. Finally, the map $z \mapsto Z_1 = z^2$ is the squaring map, whose picture we discussed at length in class.

5.8 Let $k$ be an positive integer, and $\alpha \in \mathbb{C}$. Find all the critical points of

$$f(z) = \frac{1}{k + 1} z^{k+1} - \alpha z,$$

as well as their order.

**Hint:** the cases $\alpha = 0$ and $\alpha \neq 0$ are radically different!

**Challenge:** to see how $f(z)$ looks like some $z^m$ around a critical point $z_0$, use a computer algebra system to plot $u = u_0$ and $v = v_0$ (where $u_0 + iv_0 = w_0 = f(z_0)$) for some concrete choices of $k$ and $\alpha$. The case $k = 2$, $\alpha = -1$ actually showed up in my research a year ago: all of this is not just a purely academic exercise!
The derivative of $f(z)$ is $f'(z) = z^k - \alpha$. If $\alpha = 0$, it has a zero of order $k$ at the origin — so $f(z)$ has a critical point of order $k$ at $z = 0$. If $\alpha \neq 0$, $f'(z)$ has $k$ simple zeroes at the $k$th roots of $\alpha$ — in other words, $f(z)$ has critical points of order 1 at each of those points.

The exponential

5.9 Suppose that a function $f(z) = u(x, y) + iv(x, y)$ satisfies the following two conditions:

1. $f(x + i0) = e^x$, and
2. $f$ is entire, with derivative $f'(z) = f(z)$.

Follow the steps below to show that $f(z)$ must, in fact, be the exponential function.
(i) Obtain the equations \( u_x = u \) and \( v_x = v \) and then use them to show that there exist real-valued functions \( \phi \) and \( \psi \) of the real variable \( y \) such that

\[
u(x, y) = e^x \phi(y), \quad \text{and} \quad v(x, y) = e^x \psi(y).
\]

(ii) Use the fact that \( u \) is harmonic to obtain the differential equation \( \phi''(y) + \phi(y) = 0 \) and thus show that \( \phi(y) = A \cos y + B \sin y \), where \( A \) and \( B \) are complex numbers.

(iii) After pointing out why \( \psi(y) = A \sin y - B \cos y \) and noting that

\[
u(x, 0) + iv(x, 0) = e^x,
\]

find \( A \) and \( B \). Conclude that

\[
u(x, y) = e^x \cos y, \quad \text{and} \quad v(x, y) = e^x \sin y.
\]

(i) If the function \( f \) is entire, then \( f'(z) = u_x + iv_x \) at every point \( z \). But \( f'(z) = f(z) = u + iv \), and thus \( u_x = u \) and \( v_x = v \). Note that

\[
(u e^{-x})_x = u_x e^{-x} - u e^{-x} = u e^{-x} - u e^{-x} = 0,
\]

which implies that \( u(x, y) e^{-x} \) is independent of \( x \). Hence \( u(x, y) = e^x \phi(y) \) for some real-valued function \( \phi(y) \). Likewise, \( v(x, y) = e^x \psi(y) \) for some real-valued function \( \psi(y) \).

(ii) Since the function \( u \) is harmonic, we have

\[
u_{xx} + v_{yy} = e^x \phi(y) + e^x \phi''(y) = 0 \quad \implies \quad \phi''(y) + \phi(y) = 0.
\]

The general solution of this equation is \( \phi(y) = A \cos y + B \sin y \), with \( A \) and \( B \) real numbers. The same argument applied to \( v \) yields \( v(y) = C \cos y + D \sin y \) with \( C \) and \( D \) real.

(iii) The Cauchy-Riemann equations imply that

\[
\begin{align*}
e^x (A \cos y + B \sin y) & = u_x = v_y = e^x (-C \sin y + D \cos y), \\
& \implies \quad A \cos y + B \sin y = D \cos y - C \sin y.
\end{align*}
\]

As the functions \( \cos y \) and \( \sin y \) are linearly independent, we have that \( A = D \) and \( B = -C \), and \( \psi(y) = A \sin y - B \cos y \).

On the other hand \( u(x, 0) + iv(x, 0) = e^x \), so

\[
u(x, 0) = e^x \phi(0) = e^x (A \cos 0 + B \sin 0) = Ae^x = e^x
\]

and

\[
v(x, 0) = e^x \psi(0) = e^x (A \sin 0 - B \cos 0) = -Be^x = 0,
\]

which imply that \( A = 1 \) and \( B = 0 \); therefore \( u(x, y) = e^x \cos x \) and \( v(x, y) = e^x \sin x \).

5.10 If \( w = e^z \) is purely imaginary, what restriction is placed on \( z \)? In other words, what is the inverse image by the exponential function of the imaginary axis \( u = 0 \)?
If $e^z = e^x (\cos y + i \sin y)$ is purely imaginary, then $e^x \cos y = 0$. Since $e^x \neq 0$, it has to be that $\cos y = 0$. Hence the complex numbers for which $e^z$ is purely imaginary are those of the form 

$$z = x + \frac{(2k+1)\pi i}{2}, \quad x \in \mathbb{R}, \ k \in \mathbb{Z}.$$ 

5.11 Use the very definition of the exponential function to show that $\overline{e^z} = e^{\overline{z}}$.

**Note:** this is a simple algebraic proof, but it has a clear geometric interpretation: reflection across the real axis in the $z$-plane results in reflection across the real axis in the $w$-plane, if $w = e^z$. 

$$\overline{e^z} = e^{x+iy} = e^x \cos y + ie^x \sin y = e^x \cos y - ie^x \sin y$$

$$= e^x \cos(-y) + ie^x \sin(-y) = e^{x-iy} = e^{\overline{z}}.$$