Classification of two-dimensional Frobenius and $H^*$-algebras

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Abstract

After a quick review of Frobenius and $H^*$-algebras, I produce explicit constructions of all the two-dimensional algebras of these kinds. With an eye toward higher dimensions, I favor general techniques over elementary ones. Impatient readers can skip straight to the finished constructions in Propositions 1, 2, and 3.

1 Algebra review

1.1 Frobenius algebras

1.1.1 Definition

Let $k$ be a field. A Frobenius algebra is a $k$-vector space $A$ equipped with linear maps

that satisfy the following conditions.

- $\mu$ is an associative multiplication with unit $\eta$:  

\[ \mu \circ (\mu \circ (\mu \circ \eta \otimes A) \otimes A) = \mu \circ (\eta \otimes A) \circ (\mu \circ (A \otimes A) \otimes A) \]

(1)
• $\delta$ is a coassociative comultiplication with counit $\epsilon$:

• $\mu$ and $\delta$ are related by the Frobenius identity.

For convenience, define $\epsilon := \eta(1)$.

1.1.2 Frobenius form

Composing $\epsilon$ with $\mu$ yields a bilinear form

\[ k \]

\[ := \]

\[ \varepsilon \]

\[ \mu \]

\[ A \]

\[ A \]
called the Frobenius form. Because $\mu$ is associative,

$$\sigma(\mu(x \otimes y), z) = \sigma(x, \mu(y \otimes z))$$

for all $x, y, z \in A$. The identity

implies that the linear functional $\sigma(x, \_)$ is nonzero for every nonzero $x \in A$, because any $x$ for which $\sigma(x, \_) = 0$ would be in the kernel of the map shown above. Combining this argument with its mirror image, we see that $\sigma$ is non-degenerate.

### 1.1.3 Commutativity and cocommutativity

The twist operator

is the map that sends $v \otimes w$ to $w \otimes v$ for all $v, w \in A$.

A Frobenius algebra is said to be commutative if

$$\mu = \mu$$
Because of the way the Frobenius identity relates multiplication and comultiplication, you might suspect that a Frobenius algebra is commutative if and only if it is cocommutative. This turns out to be true. See Appendix A for a proof, based on the one in [2].

1.2 $H^*$-algebras

1.2.1 Definition

Suppose $k$ is a subfield of $\mathbb{C}$, so a vector space over $k$ can be an inner product space. In this case, an $H^*$-algebra is a Frobenius algebra equipped with an inner product that makes $\delta = \mu^\dagger$ and $\epsilon = \eta^\dagger$.

2 Classification

2.1 Assumptions about the base field

Our classification splits into two cases, which rely on different assumptions about the field $k$. When $\sigma(e,e) = 0$, we assume that $k$ does not have characteristic two. When $\sigma(e,e) \neq 0$, we assume that every element of $k$ has a square root.

2.2 Frobenius algebras

Say $A$ is a two-dimensional Frobenius algebra. Pick any $v \in A$ outside the span of $e$. The condition that $e$ is a unit for $\mu$ is satisfied if and only if

\[\mu(e \otimes e) = e \quad \mu(e \otimes v) = v \quad \mu(v \otimes e) = v.\]

Since $e \otimes e$, $e \otimes v$, $v \otimes e$, and $v \otimes v$ form a basis for $A \otimes A$, it follows that $A$ is commutative.

Because $\sigma$ is non-degenerate, the kernel of $\sigma(e, \_)$ is one-dimensional. Here, our classification splits into two parts: the case where $e$ is in the kernel of $\sigma(e, \_)$ and the case where it isn’t.

\[^1\text{To make sense of this definition, remember that the tensor product of two inner product spaces $V$ and $W$ has a canonical inner product, defined by the equation } \langle v \otimes w, \tilde{v} \otimes \tilde{w} \rangle = \langle v, \tilde{v} \rangle \langle w, \tilde{w} \rangle.\]
2.2.1 Null unit

Suppose \( \sigma(e,e) = 0 \), and \( k \) does not have characteristic two. Since \( A \) is commutative, \( \sigma \) is symmetric. The subspace spanned by \( e \) is Lagrangian, and \( k \) does not have characteristic two, so we can find an element \( x \) with \( \sigma(x,x) = 0 \) whose span is complementary to the span of \( e \) (see Appendix B for details). We can assume without loss of generality that \( \sigma(e,x) = 1 \). The elements \( e \) and \( x \) form a basis for \( A \).

Observe that

\[
\epsilon(e) = \epsilon \mu(e \otimes e) = \sigma(e,e) = 0 \\
\epsilon(x) = \epsilon \mu(e \otimes x) = \sigma(e,x) = 1.
\]

Once \( \mu(x \otimes x) \) is fixed, the action of \( \mu \) is uniquely determined by the condition that \( e \) is a unit for \( \mu \). Write \( \mu(x \otimes x) \) as a linear combination \( pe + qx \). Observe that

\[
\sigma(x,x) = \epsilon \mu(x \otimes x) = \epsilon(p e + q x) = q.
\]

By construction, \( \sigma(x,x) = 0 \), so \( q = 0 \). Thus, the only degree of freedom for \( \mu \) is the value of \( p \) in the equation

\[
\mu(x \otimes x) = pe.
\]

Write \( \delta(e) \) and \( \delta(x) \) as linear combinations

\[
\delta(e) = ae \otimes e + b(e \otimes x + x \otimes e) + c x \otimes x \\
\delta(x) = \tilde{a} e \otimes e + \tilde{b}(e \otimes x + x \otimes e) + \tilde{c} x \otimes x.
\]

Since

\[
[\epsilon \otimes \text{id}] \delta(e) = be + cx \\
[\epsilon \otimes \text{id}] \delta(x) = \tilde{b}e + \tilde{c}x,
\]

the condition that \( \epsilon \) is a counit for \( \delta \) is satisfied if and only if

\[
b = 1 \\
c = 0 \\
\tilde{b} = 0 \\
\tilde{c} = 1.
\]

To see when the Frobenius identity is satisfied, let’s make tables of values for \([\text{id} \otimes \mu][\delta \otimes \text{id}]\) and \(\delta \mu\). Because \( e \) is a unit for \( \mu \), the Frobenius identity is guaranteed to hold for the inputs \( e \otimes e \) and \( x \otimes e \), so we can omit these from our tables.

\[
\begin{array}{ccc}
v & [\delta \otimes \text{id}](v) & [\text{id} \otimes \mu][\delta \otimes \text{id}](v) \\
\hline
e \otimes e & ae \otimes e + e \otimes x + x \otimes e \otimes x & ae \otimes x + e \otimes pe + x \otimes x \\
x \otimes x & \tilde{a} e \otimes e + e \otimes x \otimes x & \tilde{a} e \otimes x + x \otimes pe \\
\end{array}
\]
Comparing the tables, we see that the Frobenius identity is satisfied if and only if \( a = 0 \) and \( \tilde{a} = p \).

It is straightforward to verify that \( \mu \) and \( \delta \) are associative and coassociative, respectively, for any value of \( p \). Once that is done, we have proven the following classification result.

**Proposition 1.** Let \( k \) be a field that does not have characteristic two. Let \( A \) be a two-dimensional \( k \)-vector space with basis \( \{ e, x \} \), and let \( \eta \colon k \to A \) be the map \( 1 \mapsto e \).

For any \( p \in k \), the linear maps defined by the equations

\[
\begin{align*}
\epsilon(e) &= 0 \\
\epsilon(x) &= 1 \\
\mu(e \otimes e) &= e \\
\mu(x \otimes e) &= x \\
\mu(e \otimes x) &= x \\
\mu(x \otimes x) &= pe \\
\delta(e) &= e \otimes x + x \otimes e \\
\delta(x) &= pe \otimes e + x \otimes x
\end{align*}
\]

make \( A \) into a Frobenius algebra. In fact, every two-dimensional Frobenius algebra over \( k \) with \( \sigma(e, e) = 0 \) is of this form.

**2.2.2 Non-null unit**

Suppose \( \sigma(e, e) \neq 0 \), and every element of \( k \) has a square root. For convenience, define \( m := \sigma(e, e) \). Pick an element \( x \) that spans the kernel of \( \sigma(e, \_ \) \). Since every element of \( k \) has a square root, we can assume without loss of generality that \( \sigma(x, x) = m \). The elements \( e \) and \( x \) form a basis for \( A \).

Since

\[
\begin{align*}
\epsilon(e) &= \epsilon(\mu(e \otimes e)) = \sigma(e, e) = m \\
\epsilon(x) &= \epsilon(\mu(e \otimes x)) = \sigma(e, x) = 0,
\end{align*}
\]

the action of \( \epsilon \) depends only on \( m \).

Once \( \mu(x \otimes x) \) is fixed, the action of \( \mu \) is uniquely determined by the condition that \( e \) is a unit for \( \mu \). Write \( \mu(x \otimes x) \) as a linear combination \( pe + qx \). Observe that

\[
\begin{align*}
\sigma(x, x) &= \epsilon(\mu(x \otimes x)) \\
&= \epsilon(\sigma(e, x)) \\
&= pe(e) + qe(x) \\
&= pm.
\end{align*}
\]
By construction, \( \sigma(x, x) = m \), so \( p = 1 \). Thus, the only degree of freedom for \( \mu \) is the value of \( q \) in the equation

\[
\mu(x \otimes x) = e + qx.
\]

Write \( \delta(e) \) and \( \delta(x) \) as linear combinations

\[
\delta(e) = ae \otimes e + b(e \otimes x + x \otimes e) + cx \otimes x
\]

\[
\delta(x) = \tilde{a}e \otimes e + \tilde{b}(e \otimes x + x \otimes e) + \tilde{c}x \otimes x.
\]

Since

\[
[e \otimes \text{id}]\delta(e) = ame + bmx
\]

\[
[e \otimes \text{id}]\delta(x) = \tilde{a}me + \tilde{b}mx,
\]

the condition that \( e \) is a counit for \( \delta \) is satisfied if and only if

\[
a = \frac{1}{m} \quad b = 0 \quad \tilde{a} = 0 \quad \tilde{b} = \frac{1}{m}.
\]

To see when the Frobenius identity is satisfied, let’s make tables of values for \([\text{id} \otimes \mu][\delta \otimes \text{id}]\) and \(\delta \mu\). Because \( e \) is a unit for \( \mu \), the Frobenius identity is guaranteed to hold for the inputs \( e \otimes e \) and \( x \otimes e \), so we can omit these from our tables.

<table>
<thead>
<tr>
<th>(v)</th>
<th>(<a href="v">\delta \otimes \text{id}</a>)</th>
<th>([\text{id} \otimes \mu]<a href="v">\delta \otimes \text{id}</a>)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(e \otimes x)</td>
<td>(\frac{1}{m} e \otimes e \otimes x + cx \otimes x \otimes x)</td>
<td>(\frac{1}{m} e \otimes x + cx \otimes (e + qx))</td>
</tr>
<tr>
<td>(x \otimes x)</td>
<td>(\frac{1}{m} (e \otimes x + x \otimes e) \otimes x + \tilde{c}x \otimes x \otimes x)</td>
<td>(\tilde{c}x \otimes x \otimes x + \frac{1}{m} e \otimes (e + qx) + \frac{1}{m} x \otimes x \otimes (e + qx))</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(v)</th>
<th>(\delta \mu(v))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(e \otimes x)</td>
<td>(\frac{1}{m} (e \otimes x + x \otimes e) + \tilde{c}x \otimes x)</td>
</tr>
<tr>
<td>(x \otimes x)</td>
<td>(\frac{1}{m} e \otimes e + \frac{q}{m} (e \otimes x + x \otimes e) + (c + q\tilde{c}) x \otimes x)</td>
</tr>
</tbody>
</table>

Comparing the tables, we see that the Frobenius identity is satisfied if and only if \( c = \frac{1}{m} \) and \( \tilde{c} = \frac{q}{m} \).

It is straightforward to verify that \( \mu \) and \( \delta \) are associative and coassociative, respectively, for any values of \( m \) and \( q \). Once that is done, we have proven the following classification result.

**Proposition 2.** Let \( k \) be a field in which every element has a square root. Let \( A \) be a two-dimensional \( k \)-vector space with basis \( \{e, x\} \), and let \( \eta: k \rightarrow A \) be the map \( 1 \mapsto e \).
For any \( q \in k \) and any nonzero \( m \in k \), the linear maps defined by the equations

\[
\begin{align*}
\epsilon(e) &= m \\
\epsilon(x) &= 0 \\
\mu(e \otimes e) &= e \\
\mu(x \otimes e) &= x \\
\mu(e \otimes x) &= x \\
\mu(x \otimes x) &= e + qx \\
\delta(e) &= \frac{1}{m}(e \otimes e + x \otimes x) \\
\delta(x) &= \frac{1}{m}(e \otimes x + x \otimes e + qx \otimes x)
\end{align*}
\]

make \( A \) into a Frobenius algebra. In fact, every two-dimensional Frobenius algebra over \( k \) with \( \sigma(e, e) \neq 0 \) is of this form.

### 2.3 \( H^*\)-algebras

Say \( A \) is a two-dimensional \( H^*\)-algebra. Since

\[
\langle e, e \rangle = \langle \eta(1), e \rangle = \langle 1, \epsilon(e) \rangle = \epsilon(e) = \sigma(e, e),
\]

the positive definiteness of the inner product guarantees that \( \sigma(e, e) \neq 0 \), so \( A \) can be presented in the form described by Proposition 2.

We immediately see that \( \langle e, e \rangle = m \), implying that \( m \) is real and positive. In addition,

\[
\begin{align*}
\langle e, x \rangle &= \langle \eta(1), x \rangle \\
&= \langle 1, \epsilon(x) \rangle \\
&= 0.
\end{align*}
\]

Furthermore,

\[
\begin{align*}
\langle \mu(x \otimes x), e \rangle &= \langle x \otimes x, \delta(e) \rangle \\
\langle e + qx, e \rangle &= \langle x \otimes x, \frac{1}{m}(e \otimes e + x \otimes x) \rangle \\
&= \frac{1}{m} \langle x, x \rangle^2 \\
m &= \frac{1}{m} \langle x, x \rangle^2 \\
m^2 &= \langle x, x \rangle^2,
\end{align*}
\]

so \( \langle x, x \rangle = m \) by positive definiteness.

On the other hand, suppose \( A \) is a two-dimensional Frobenius algebra in the form described by Proposition 2, equipped with the conjugate-symmetric bilinear form defined by

\[
\begin{align*}
\langle e, e \rangle &= m \\
\langle e, x \rangle &= 0 \\
\langle x, x \rangle &= m.
\end{align*}
\]
Which values of $m$ and $q$ make $A$ into an $H^*$-algebra?

For $\langle \_, \_ \rangle$ to be an inner product, $m$ must be real and positive. The condition that $\epsilon = \eta^\dagger$ is always satisfied, and it is straightforward to verify that $\delta = \mu^\dagger$ if and only if $q$ is real.\footnote{The restriction on $q$ comes from the equation $\langle \mu(x \otimes x), x \rangle = \langle x \otimes x, \delta(x) \rangle$.} Once that is done, we have proven the following classification result.

**Proposition 3.** Let $A$ be a two-dimensional Frobenius algebra in the form described by Proposition 2. If $m$ is real and positive, and $q$ is real, the inner product defined by

$$
\langle e, e \rangle = m \quad \langle e, x \rangle = 0 \quad \langle x, x \rangle = m
$$

makes $A$ into an $H^*$-algebra. In fact, every two-dimensional $H^*$-algebra over $k$ is of this form.

**A  Commutativity and cocommutativity**

I will show that every commutative Frobenius algebra is cocommutative, following the proof in [2]. To get the converse, turn the argument upside down.

I will assume several identities involving the twist operator. Some of them are rather subtle, so be careful to think about why they are true.

Suppose $A$ is commutative. We get our foot in the door by showing that the “twisted coproduct” satisfies part of the Frobenius identity (Figure 1). Then we observe that

![Diagram](image)

Applying these identities in just the right way, we can untwist the twisted coproduct (Figure 2).

**B  Lagrangian complements**

Assume $k$ does not have characteristic two.

Say $V$ is a $2n$-dimensional vector space equipped with a $t$-symmetric,\footnote{Symmetric if $t = 1$, skew-symmetric if $t = -1$.} non-degenerate bilinear form $\sigma$, and $L \subset V$ is a Lagrangian subspace—an $n$-dimensional...
Figure 1: The twisted coproduct satisfies part of the Frobenius identity.
Figure 2: Untwisting the twisted coproduct.
subspace with $\sigma(\ell, \ell) = 0$ for all $\ell \in L$. Following [1, Proposition 8.2], I will give a procedure for turning any subspace $W \subset V$ complementary to $L$ into a Lagrangian subspace of $V$ complementary to $L$.

Let $S : V \to V'$ be the map $v \mapsto \sigma(v, \cdot)$. Since $\sigma$ is non-degenerate, $S$ is an isomorphism.

Let $\pi : V' \to W'$ be the dual of the inclusion $W \hookrightarrow V$. Observe that

$$\ker(\pi) = \{ \xi \in V' : \xi(W) = 0 \},$$

so

$$\ker(\pi S) = \{ v \in V : S(v)(W) = 0 \}.$$

Being the dual of an injection, $\pi$ is surjective, so

$$\dim \ker(\pi) = \dim V' - \dim W' = n.$$

Since $S$ is an isomorphism, $\ker(\pi S)$ has dimension $n$ as well.

Any vector $\ell$ in the intersection of $\ker(\pi S)$ and $L$ must have $S(\ell)(W) = 0$ and $S(\ell)(L) = 0$. Since $W$ and $L$ are complements, it follows that $S(\ell) = 0$, which means $\ell = 0$. Therefore, $\ker(\pi S)$ intersects $L$ only at zero.

Think of $V$ as $W \oplus L$. Since $\ker(\pi S)$ intersects $L$ only at zero, it is the graph of a linear map $F : W \to L$. Notice that for any $w, \tilde{w} \in W$,

$$S(w + Fw)(\tilde{w}) = 0,$$

because $w + Fw \in \ker(\pi S)$. Therefore,

$$\sigma(Fw, \tilde{w}) = -\sigma(w, \tilde{w})$$

for all $w, \tilde{w} \in W$.

I claim that the graph of $\frac{1}{2}F$ is a Lagrangian subspace of $V$ complementary to $L$. To see why, pick any $w, \tilde{w} \in W$, and observe that

$$\sigma(w + \frac{1}{2}Fw, \tilde{w} + \frac{1}{2}F\tilde{w}) = \sigma(w, \tilde{w}) + \frac{1}{2}\sigma(w, F\tilde{w}) + \frac{1}{2}\sigma(Fw, \tilde{w}) + \frac{1}{4}\sigma(Fw, F\tilde{w})$$

$$= \sigma(w, \tilde{w}) + \frac{1}{2}t\sigma(F\tilde{w}, w) + \frac{1}{2}\sigma(Fw, \tilde{w}) + 0$$

$$= \sigma(w, \tilde{w}) - \frac{1}{2}t\sigma(\tilde{w}, w) - \frac{1}{2}\sigma(w, \tilde{w})$$

$$= \sigma(w, \tilde{w}) - \frac{1}{2}\sigma(w, \tilde{w}) - \frac{1}{2}\sigma(w, \tilde{w})$$

$$= 0.$$

The graph of $F$ has dimension $n$, so the graph of $\frac{1}{2}F$ also has dimension $n$. Therefore, the graph of $\frac{1}{2}F$ is a Lagrangian subspace of $V$. Since $\frac{1}{2}F$ is a map to $L$, its graph is complementary to $L$.

References
