Farey Sets in \( \mathbb{R}^n \)

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The Farey numbers of order \( Q \) are the fractions between zero and one whose denominators are less than or equal to \( Q \). You can think of these numbers as the intersection of the interval \([0, 1]\) with the set

\[
\mathcal{F}_Q = \bigcup_{q=1}^Q \frac{1}{q} \mathbb{Z},
\]

where \( \frac{1}{q} \mathbb{Z} \) is shorthand for \( \{ \frac{p}{q} \mid p \in \mathbb{Z} \} \).

An obvious analogue of \( \mathcal{F}_Q \) in \( \mathbb{R}^n \) is

\[
n\mathcal{F}_Q = \bigcup_{q=1}^Q \frac{1}{q} \mathbb{Z}^n.
\]

Look at the plots of \( ^1\mathcal{F}_Q \) and \( ^2\mathcal{F}_Q \) in Figures 1 and 2. What’s up with those empty regions? It turns out that in \( ^n\mathcal{F}_Q \), if you pick a lattice point \( a \in \mathbb{Z}^n \) and a fraction \( r/s \) in lowest terms, the hyperplane

\[
a \cdot x = r/s
\]

is sandwiched between empty regions of width slightly greater than

\[
\frac{\text{gcf}(a)}{Qs\|a\|},
\]

with “slightly greater than” going to zero as \( Q \) goes to infinity. Here, \( \cdot \) is the standard inner product on \( \mathbb{R}^n \), \( \text{gcf}(a) \) is shorthand for \( \text{gcf}(a_1, \ldots, a_n) \), and \( \|a\| = \sqrt{a \cdot \bar{a}} \).

The observation above is a fairly straightforward consequence of the following two facts.

**Fact 1.** If you project \( ^n\mathcal{F}_Q \) onto the line generated by \( a \in \mathbb{Z}^n \), which is isometric to \( \mathbb{R} \), you end up with

\[
\frac{\text{gcf}(a)}{\|a\|} \mathcal{F}_Q.
\]

Figure 1: A plot of \( ^1\mathcal{F}_{16} \) on the interval \([-1, 1]\).
Figure 2: A plot of $^2\mathcal{F}_{40}$ in the box $[-1, 1]^2$. 
Fact 2. If the fraction \( r/s \) is in lowest terms, the distances between \( r/s \) and its neighbors in \( 1 \mathcal{F}_Q \) are equal to or slightly greater than \( 1/Qs \), with "slightly greater than" going to zero as \( Q \) goes to infinity.

Proof of Fact 1. Since

\[
\mathcal{F}_Q = \bigcup_{q=1}^{Q} \frac{1}{q} \mathbb{Z}^n,
\]

the projection of \( \mathcal{F}_Q \) onto the line generated by \( a \in \mathbb{Z}^n \) is

\[
\bigcup_{q=1}^{Q} \frac{1}{|a|} a \mathbb{Z}^n,
\]

where \( a \cdot \mathbb{Z}^n \) is shorthand for

\[
\{a \cdot z | z \in \mathbb{Z}^n\} = \{a_1 z_1 + \ldots + a_n z_n | z_1, \ldots, z_n \in \mathbb{Z}\}.
\]

By Bézout’s identity,

\[
\{a_1 z_1 + \ldots + a_n z_n | z_1, \ldots, z_n \in \mathbb{Z}\} = \{\gcd(a_1, \ldots, a_n) z | z \in \mathbb{Z}\},
\]

in shorthand,

\[
a \cdot \mathbb{Z}^n = \gcd(a) \mathbb{Z}.
\]

Therefore, the projection of \( \mathcal{F}_Q \) onto the line generated by \( a \in \mathbb{Z}^n \) is

\[
\bigcup_{q=1}^{Q} \frac{1}{|a|} \gcd(a) \mathbb{Z} = \gcd(a) \bigcup_{q=1}^{Q} \frac{1}{q} \mathbb{Z} = \gcd(a) \mathcal{F}_Q.
\]

\( \square \)

Proof of Fact 2. Since the elements of \( \mathcal{F}_Q \) are rational numbers, we can put them in increasing order, and we can also write them as fractions in lowest terms. In this proof, I’ll think of the \( \mathcal{F}_Q \) not as sets of rational numbers, but as increasing sequences of fractions in lowest terms.

We know from the work of Charles Haros, and many others who followed him,\(^1\) that you can turn \( \mathcal{F}_Q \) into \( \mathcal{F}_Q \) by following a simple rule:

If you see two adjacent fractions \( \frac{a}{b} \) and \( \frac{c}{d} \) whose denominators add up to \( Q \), insert their mediant \( \frac{a+c}{b+d} \) between them.

Starting with \( \mathcal{F}_1 \), you can generate \( \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4 \ldots \) by using this rule over and over. If the fraction \( r/s \) is in lowest terms, it first appears in \( \mathcal{F}_1 \), as the mediant of two fractions \( a/b \) and \( c/d \), with

\[
\frac{a}{b} < \frac{r}{s} < \frac{c}{d}.
\]

\(^1\)For details, I recommend the excellent book *A Motif of Mathematics*, by Scott Guthery.
The fraction $a/b$ is the lower neighbor of $r/s$ until you reach $1_{F_{b+s}}$, where a new fraction appears between $a/b$ and $r/s$:

$$\frac{a+2r}{b+2s}.$$ 

This fraction remains the lower neighbor of $r/s$ until it is displaced, in $1_{F_{b+2s}}$, by 

$$\frac{a+2r}{b+2s}.$$ 

In general, the lower neighbor of $r/s$ in $1_{F_{b+ms}}$ is 

$$\frac{a+mr}{b+ms}.$$ 

Similarly, the upper neighbor of $r/s$ in $1_{F_{ns+d}}$ is 

$$\frac{nr+c}{ns+d}.$$ 

Because $a/b$ is the lower neighbor of $r/s$ in one of the $1_{F_Q}$, we have the identity $rb - sa = 1$, which you can easily prove by induction. Hence, the distance between $r/s$ and its lower neighbor in $1_{F_{b+ms}}$ is 

$$\frac{r - a + mr}{s} - \frac{r - sa}{b + ms} = \frac{rb - sa}{s(b + ms)} = \frac{1}{s(b + ms)}.$$ 

Similarly, from the identity $cs - dr$, we find that the distance between $r/s$ and its upper neighbor in $1_{F_{ns+d}}$ is 

$$\frac{nr + c}{ns + d} - \frac{r}{s} = \frac{1}{s(ns + d)}.$$ 

Now, for any $Q \geq s$, pick the largest $m$ so that $b + ms \leq Q$, and the largest $n$ so that $ns + d \leq Q$. The distances between $r/s$ and its neighbors in $1_{F_Q}$ are 

$$\frac{1}{s(b + ms)}$$ 

and 

$$\frac{1}{s(ns + d)},$$ 

respectively. Both distances are equal to or slightly greater than $1/Qs$, and as $Q$ goes to infinity, $b + ms$ and $ns + d$ approach $Q$. 

\[\Box\]