Week 5 notes

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Chaos, fractals, and dynamics
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1 The formal definition of semiconjugacy

Last time, we introduced the idea of semiconjugacy through an example: the binary representation of angles, which relates the doubling map to the shift map. Today, we'll define semiconjugacy formally, and see a second example of a semiconjugacy.

1.1 Review of the binary representation

First, let's recall last week's example of a semiconjugacy, so we can have it in mind when we see how semiconjugacy is defined. The binary representation of angles is the function $\phi\colon \mathbf{2}^{\mathbb{N}} \to \mathbb{T}$ defined by

$$\phi(w) \equiv 2\pi(\text{the number with binary digit sequence } w).$$

This function creates a useful relationship between the doubling map $D\colon \mathbb{T} \to \mathbb{T}$ and the shift map $S\colon \mathbf{2}^{\mathbb{N}} \to \mathbf{2}^{\mathbb{N}}$. When you double a number, each digit of its binary representation moves one place to the left, so we can find out what the doubling map does to an angle by looking at what the shift map does to its binary representation. The formula

$$D(\phi(w)) = \phi(S(w))$$

and the picture

$$\begin{array}{ccc} \mathbf{2}^{\mathbb{N}} & \xrightarrow{S} & \mathbf{2}^{\mathbb{N}} \\ {\scriptstyle\phi}\downarrow & /\!/ & \downarrow{\scriptstyle\phi} \\ \mathbb{T} & \xrightarrow[D]{} & \mathbb{T} \end{array}$$

express this relationship.

1.2 The definition of semiconjugacy

Let's try to distill the essential features of this situation.

- We've got a useful relationship between a dynamical map $F\colon X \to X$ and another dynamical map $E\colon W \to W$.

  (In our example, these are the doubling map $D\colon \mathbb{T} \to \mathbb{T}$ and the shift map $S\colon \mathbf{2}^{\mathbb{N}} \to \mathbf{2}^{\mathbb{N}}$.)

- We created the relationship by treating the points in W as "labels" for the points in X. There's a function $\psi\colon W \to X$ that sends each label to the point it stands for.

  (In our example, this is the binary representation $\phi\colon \mathbf{2}^{\mathbb{N}} \to \mathbb{T}$.)

Our labeling is useful because it has the following properties.

- We can find out what F does to a point by looking at what E does to its label. In symbols,
$$F(\psi(w)) = \psi(E(w)) \quad \text{for all labels } w \in W.$$
In a picture,
$$\begin{array}{ccc} W & \xrightarrow{E} & W \\ {\scriptstyle\psi}\downarrow & /\!/ & \downarrow{\scriptstyle\psi} \\ X & \xrightarrow[F]{} & X \end{array}$$

- Every point in X has a label. In other words, every point $x \in X$ can be expressed as $\psi(w)$ for some $w \in W$. A function ψ with this property is called *onto*.

- Each point in X has a limited number of labels. Specifically, we can fix a maximum m and say that that each point in X has at most m labels in W. A function ψ with this property is called *at most m-to-one*.

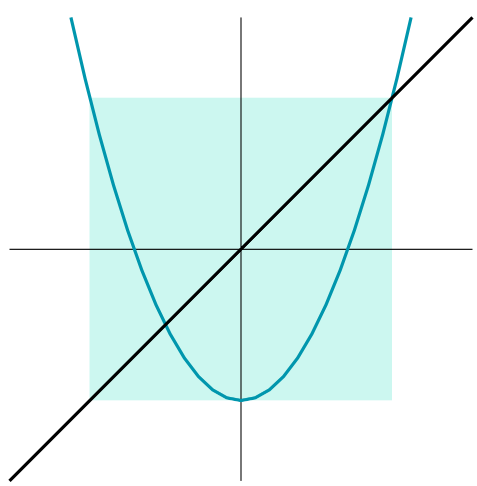
- The function ψ is *continuous*.

  (If the state spaces X and W resemble the real line, you know what this means. For weirder-looking state spaces, we need to generalize the idea of a continuous function, just like we generalized open intervals and limits earlier. We'll talk about that soon.)

A function ψ with these properties is called a *semiconjugacy* from E to F.

1.3 A second example: a quadratic map and the doubling map

Let's try out our definition of semiconjugacy by looking at a second example. Consider the quadratic map $F(x) = x^2 - 2$. When we apply F, every point in the interval $[-2, 2]$ stays inside that interval, so we can use $[-2, 2]$ as our state space for F instead of the whole real line. Let's write
$$F\colon [-2, 2] \to [-2, 2]$$
to remind ourselves that we're using a different state space than usual.



If you ponder this graph long enough, you should be able to see that F takes the interval $[-2, 2]$, folds it in half, and then stretches it back to full length. Another word for folding in half is "doubling"—and there turns out to be a semiconjugacy from the doubling map to F!

We can treat points on the unit circle as labels for points in $[-2, 2]$ using a trig function. Let's try the function $\psi\colon \mathbb{T} \to [-2, 2]$ given by the formula $\psi(\theta) = 2\cos(\theta)$. To see whether ψ is a semiconjugacy from the doubling map to F, there are three properties we need to check.

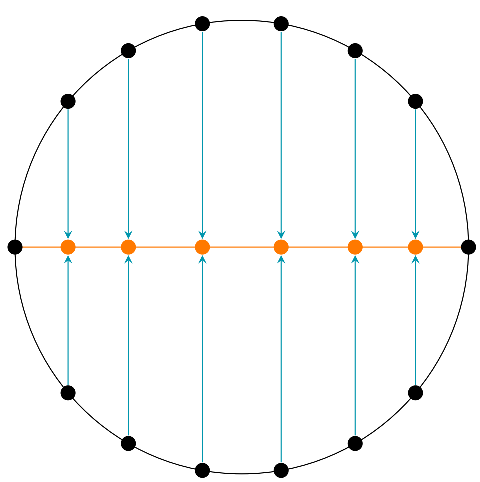
- Can we find out what F does to a point by looking at what D does to its label? In symbols:

$$\begin{aligned} F(\psi(\theta)) &= \psi(D(\theta)) && ? \\ F(2\cos(\theta)) &= 2\cos D(\theta) && ? \\ (2\cos(\theta))^2 - 2 &= 2\cos 2\theta && ? \end{aligned}$$

  Using the angle addition formula, we get

$$\begin{aligned} 4\cos(\theta)^2 - 2 &= 2[\cos(\theta)^2 - \sin(\theta)^2] && ? \\ 4\cos(\theta)^2 - 2 &= 2[\cos(\theta)^2 - (1 - \cos(\theta)^2)] && ? \\ 2[2\cos(\theta)^2 - 1] &= 2[2\cos(\theta)^2 - 1] && ✔ \end{aligned}$$

- Does each point in $[-2, 2]$ have a label? Yes, as you can see from this picture of how labels on the circle correspond to points in $[-2, 2]$.



- Does each point in $[-2, 2]$ have a limited number of labels? Yes: you can see from the picture above that ψ is at most 2-to-one.

- Is ψ continuous? Yes: the cosine function is continuous, and it stays continuous when you rescale it by a factor of two.

We've now confirmed that $\psi\colon \mathbb{T} \to [-2, 2]$ is a semiconjugacy from the doubling map $D\colon \mathbb{T} \to \mathbb{T}$ to the quadratic map $F\colon [-2, 2] \to [-2, 2]$.

2 Generalizing continuity

The binary representation $\phi\colon \mathbf{2}^{\mathbb{N}} \to \mathbb{T}$ is supposed to be a semiconjugacy from the shift map to the doubling map. We should confirm that it actually qualifies as a semiconjugacy, according to our formal definition. Before we can do that, we have to fill in a detail I left out earlier. We have to generalize the idea of continuity to functions between weird-looking state spaces, like $\mathbf{2}^{\mathbb{N}}$.

Let's start with an informal idea of what continuity means.

Informal definition. A function $\psi\colon W \to X$ is *continuous* if you can control its output as precisely as you want by controlling its input precisely enough.

"Precise control" means getting something close to where you want it. In other words, precision is defined in terms of distance, which we measure using a metric. So, to make our definition of continuity more formal, let's assume W and X both come with metrics.

Formal definition. Consider a function $\psi\colon W \to X$. Pick any point $w \in W$. The function ψ is *continuous at w* if we can keep the output of ψ within any "target" open ball around $\psi(w)$ by keeping its input within a small enough open ball around w.

The function ψ is *continuous* if it's continuous at every point in W.

2.1 Example: the squaring function

As an example, consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by the formula $f(t) = t^2$. Let's confirm that f is continuous at the point 2. Suppose we keep the input of f within the open ball of radius r around 2. Then the output of f will land in the interval $((2-r)^2, (2+r)^2)$, which we can rewrite as

$$(4 - 2r + r^2, 4 + 2r + r^2).$$

It's apparent from this formula that we can get the output of f within any open ball around 4 by making r small enough. Therefore, f is continuous at 2.

2.2 Non-example: a step function

For an example of a function that's not continuous, consider the function $g\colon \mathbb{R} \to \mathbb{R}$ defined by the formula

$$g(t) = \begin{cases} 0 & t \le 0 \\ 1 & t > 0. \end{cases}$$

Looking at the graph of g, we can guess that g is not continuous at 0. Let's confirm that guess. Suppose we keep the input of g within the open ball of radius r around 0. This ball contains both negative and positive numbers, so the possible outputs of g will include both 0 and 1. That means we can never keep the output of g within the open ball of radius $\frac{1}{2}$ around 0, no matter how small we make r. Therefore, g is not continuous at 0.

2.3 Example: the binary representation

Now for the example we've been waiting for! Let's show that the binary representation $\phi\colon \mathbf{2}^{\mathbb{N}} \to \mathbb{T}$ is continuous.

Pick any point $w \in \mathbf{2}^{\mathbb{N}}$. Using $w_1, w_2, w_3, w_4, \ldots$ to denote the digits of w, we can write

$$\phi(w) \equiv 2\pi \left(\frac{w_1}{2^1} + \frac{w_2}{2^2} + \frac{w_3}{2^3} + \frac{w_4}{2^4} + \ldots \right).$$

Let's pick an input u and see how far the output $\phi(u)$ is from $\phi(w)$. Using $u_1, u_2, u_3, u_4, \ldots$ to denote the digits of u, we can reason that

$$\begin{aligned} |\phi(u) - \phi(w)| &= 2\pi \left| \frac{u_1 - w_1}{2^1} + \frac{u_2 - w_2}{2^2} + \frac{u_3 - w_3}{2^3} + \frac{u_4 - w_4}{2^4} + \ldots \right| \\ &\le 2\pi \left(\frac{|u_1 - w_1|}{2^1} + \frac{|u_2 - w_2|}{2^2} + \frac{|u_3 - w_3|}{2^3} + \frac{|u_4 - w_4|}{2^4} + \ldots \right). \end{aligned}$$

Now, suppose we keep the input u within the open ball of radius 2^{-n} around w. Then u must match w for at least the first $n+1$ digits, so the first $n+1$ terms of the sum we just wrote must be zero. So, when $u \in B_w(2^{-n})$,

$$\begin{aligned} |\phi(u) - \phi(w)| &\le 2\pi \left(\frac{0}{2^1} + \ldots + \frac{0}{2^{n+1}} + \frac{|u_{n+2} - w_{n+2}|}{2^{n+2}} + \frac{|u_{n+3} - w_{n+3}|}{2^{n+3}} + \ldots \right) \\ &\le \frac{2\pi}{2^{n+2}} \left(\frac{|u_{n+2} - w_{n+2}|}{2^0} + \frac{|u_{n+3} - w_{n+3}|}{2^1} + \frac{|u_{n+4} - w_{n+4}|}{2^2} + \ldots \right). \end{aligned}$$

The difference between the digits u_k and w_k is at most one, of course, so

$$\begin{aligned} |\phi(u) - \phi(w)| &\le \frac{2\pi}{2^{n+2}} \left(\frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \ldots \right) \\ &= \frac{2\pi}{2^{n+2}} \left(\frac{1}{1 - \frac{1}{2}} \right) \\ &= \frac{2\pi}{2^{n+2}} \cdot 2 \\ &= \frac{2\pi}{2^{n+1}}. \end{aligned}$$

It's apparent from this formula that we can get $\phi(u)$ within any open ball around $\phi(w)$ by keeping u within a small enough ball $B_w(2^{-n})$. Therefore, ϕ is continuous at w.

Since w could have been any point in $\mathbf{2}^{\mathbb{N}}$, we've shown that ϕ is continuous.

2.4 The binary representation is a semiconjugacy

[If you're short on time, skip the details—not that there are many details in the first place.]

Now that we know the binary representation is continuous, we can finally confirm that it's a semiconjugacy from the shift map $S\colon \mathbf{2}^{\mathbb{N}} \to \mathbf{2}^{\mathbb{N}}$ to the doubling map $D\colon \mathbb{T} \to \mathbb{T}$.

- Can we find out what D does to a point by looking at what S does to its label? Yes. As we saw last Thursday, this is because doubling a number shifts each digit of its binary representation one place to the left.

- Does each point in $\mathbb{T}$ have a label? Yes. We can express any angle in the form $2\pi t$, with $t \in [0, 1]$, and every number in $[0, 1]$ has a binary representation

$$0.\,\text{string of binary digits}$$

- Does each point in $\mathbb{T}$ have a limited number of labels? Yes. For most values of t, the angle $2\pi t$ has just one binary representation. The only exception is when t is a fraction whose denominator is a power of two. In that case, the angle $2\pi t$ has two binary representations: one ending with $\overline{\mathtt{0}}$ and one ending with $\overline{\mathtt{1}}$. Therefore, ϕ is at most two-to-one.

  We saw some examples of angles with two binary representations last week:

$$\begin{aligned}
2\pi(0.\overline{0}_{\text{ binary}}) &\equiv 0 \equiv 2\pi \equiv 2\pi(0.\overline{1}_{\text{ binary}}) \\
2\pi(0.1\overline{0}_{\text{ binary}}) &\equiv \frac{2\pi}{2} \equiv 2\pi(0.0\overline{1}_{\text{ binary}}) \\
2\pi(0.01\overline{0}_{\text{ binary}}) &\equiv \frac{2\pi}{4} \equiv 2\pi(0.00\overline{1}_{\text{ binary}}) \\
2\pi(0.11\overline{0}_{\text{ binary}}) &\equiv \frac{2\pi \cdot 3}{4} \equiv 2\pi(0.10\overline{1}_{\text{ binary}})
\end{aligned}$$

- Is ϕ continuous? Yes. We checked that earlier today.

We've now confirmed that $\phi\colon \mathbf{2}^{\mathbb{N}} \to \mathbb{T}$ is a semiconjugacy from the shift map $S\colon \mathbf{2}^{\mathbb{N}} \to \mathbf{2}^{\mathbb{N}}$ to the doubling map $D\colon \mathbb{T} \to \mathbb{T}$.

3 Semiconjugacy toolbox

Last week, we used the binary representation to find lots of periodic and eventually fixed points for the doubling map. Now that we have a formal definition of semiconjugacy, we can generalize those tricks to other dynamical systems. We can also start inventing new ways of using semiconjugacy to study orbits. As we start filling up our semiconjugacy toolbox, think about how the four defining properties of a semiconjugacy work together inside of each new tool.

For the rest of this section, suppose $\psi\colon W \to X$ is a semiconjugacy from $E\colon W \to W$ to $F\colon X \to X$. Keep in mind that, according to our definition, ψ has to be at most m-to-one for some m.

3.1 Finding fixed, eventually fixed, and periodic points

3.1.1 Going forward

Let's start by explaining and generalizing the tricks we used last week to find fixed, eventually fixed, and periodic points of the doubling map.

Fact. *If w is a fixed point of E, then $\psi(w)$ is a fixed point of F.*

The explanation is the same as the one we saw for the doubling map last week.

Explanation. Suppose $E(w) = w$. Then $\psi(E(w)) = \psi(w)$. According to the definition of a semiconjugacy,

$$\begin{aligned} F(\psi(w)) &= \psi(E(w)) \\ &= \psi(w). \end{aligned}$$

. □

To explain the tricks for finding periodic and eventually fixed points, here's a fact that will come in handy.

Fact. *We have*

$$F^n(\psi(w)) = \psi(E^n(w))$$

for all labels $w \in W$ and all numbers $n \in \{0, 1, 2, \ldots\}$.

Explanation. This is easiest to see with a picture.

$$\begin{array}{ccccccccc} W & \xrightarrow{E} & W & \xrightarrow{E} & \ldots & \xrightarrow{E} & W & \xrightarrow{E} & W \\ \psi\downarrow & /\!/ & \psi\downarrow & /\!/ & & /\!/ & \psi\downarrow & /\!/ & \psi\downarrow \\ X & \xrightarrow[F]{} & X & \xrightarrow[F]{} & \ldots & \xrightarrow[F]{} & X & \xrightarrow[F]{} & X \end{array}$$

Since both paths around each square give the same result, every path along the whole rectangle gives the same result. In particular, going all the way along the the top of the rectangle and then stepping down is the same as stepping down and then going all the way along the bottom of the rectangle. □

From the fact above, you can deduce a fancier-sounding one.

Fact (♡). *Because ψ is a semiconjugacy from E to F, it's also a semiconjugacy from E^n to F^n, for any number n of iterations.*

I'll skip the explanation, because it's very straightforward. You should think about it.

Now we can explain our tricks for finding periodic and eventually fixed points.

Fact. *If w is an n-periodic point of E, then $\psi(w)$ is an n-periodic point of F.*

Explanation. Suppose $E^n(w) = w$. Using this and Fact ♡,

$$\begin{aligned} F^n(\psi(w)) &= \psi(E^n(w)) \\ &= \psi(w). \end{aligned}$$

□

Fact. *If w is an eventually fixed point of E, then $\psi(w)$ is an eventually fixed point of F.*

Explanation. Suppose $E^n(w)$ is a fixed point of E. Then $E^{n+1}(w) = E^n(w)$. Using this and Fact ♡,

$$\begin{aligned} F^{n+1}(\psi(w)) &= \psi(E^{n+1}(w)) \\ &= \psi(E^n(w)). \end{aligned}$$

On the other hand, Fact ♡ also tells us that

$$F^n(\psi(w)) = \psi(E^n(w)).$$

Putting those together, we see that

$$F^{n+1}(\psi(w)) = F^n(\psi(w)).$$

□

3.1.2 Going backward

I claimed last week that the periodic and eventually fixed points of the shift map actually label *all* of the periodic and eventually fixed points of the doubling map.

Fact. *Suppose ψ is at most m-to-one. If $\psi(w)$ is a fixed point of F, the orbit of w must eventually reach a periodic point of E, with a minimum period of at most m.*

Explanation. Using Fact ♡, you should be able to convince yourself that every point in the orbit of w is a label for $\psi(w)$. Since ψ is at most m-to-one, $\psi(p)$ has at most m labels. That means the orbit of w can't go for more than m steps without repeating itself. In other words, we can find numbers $n \in \mathbb{N}$ and $k \in \{0, \ldots, m\}$ with $E^n(w) = E^{n+k}(w)$. We can rewrite this as $E^k(E^n(w)) = E^n(w)$, showing that $E^n(w)$ is a k-periodic point. Since k is at most m, and the minimum period of $E^n(w)$ is at most k, this is what we wanted to show. □