Week 3 notes

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Chaos, fractals, and dynamics
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1 Measuring distance in a general state space

1.1 Introduction

Last week, we defined attracting and repelling fixed points and orbits for dynamical systems with state space $\mathbb{R}$. This week, we'll generalize these definitions to other state spaces.

Key concepts we used in our definitions include:

- Open intervals
- Limits

Both can be based on the concept of distance.[1] Our first goal for today is to generalize the idea of distance to other state spaces.

1.2 Examples: distance functions on $\mathbb{R}$, $\mathbb{T}$, and $\mathbf{2}^{\mathbb{N}}$

1.2.1 The standard distance function on $\mathbb{R}$

Let's start with something familiar. The distance between two points $x, y \in \mathbb{R}$ is given by the function $d(x, y) = |y - x|$.

Here's another way to express this distance function, which will sometimes come in handy. If you start at the point $x \in \mathbb{R}$ and then move by a displacement of $a \in \mathbb{R}$, the distance to the place you end up is $|a|$. In other words, $d(x, x + a) = |a|$.

1.2.2 The standard distance function on $\mathbb{T}$

The distance between two points on the circle is the length of the shortest path along the circle from one point to another. If you start at $\theta \in \mathbb{T}$ and then move by a displacement of of $\alpha \in \mathbb{R}$, the distance to the place you end up is $|\alpha|$, as long as you didn't move more than halfway around the circle. In other words, $d(\theta, \theta + \alpha) = |\alpha|$, as long as $\alpha \in [-\pi, \pi]$.

1.2.3 The standard distance function on $\mathbf{2}^{\mathbb{N}}$

Given two distinct sequences $x, y \in \mathbf{2}^{\mathbb{N}}$, let m be the number of digits before the first place they differ. We define $d(x, y) = 2^{-m}$. For example:

[1]They don't have to be, though. Ask me about this later if you'd like to learn more.

first difference

$x = 010100\,0\,101001\ldots$

$y = 01010\,1\,010101\ldots$

5 digits before first difference

$d(x,y) = 2^{-5} = \frac{1}{32}$

first difference

$x = 0\,11111111111\ldots$

$y = 1\,00001000000\ldots$

0 digits before first difference

$d(x,y) = 2^{-0} = 1$

(The textbook uses a different distance function on $\mathbf{2}^{\mathbb{N}}$. I think this one is easier to work with, and more commonly used.)

1.3 General properties of distance functions

The distance functions we just saw all take two points x, y in a state space Y and give back a number $d(x,y) \in [0,\infty)$. They share the following three properties.

- Distance is the same in both directions.

$$d(x,y) = d(y,x) \qquad \text{for all } x, y \in Y$$

- The distance from a point to itself is zero, and points with zero distance between them are the same.

$$x = y \iff d(x,y) = 0$$

- Direct trips are the shortest.

$$d(x,y) \leq d(x,p) + d(p,y) \qquad \text{for all } x, y, p \in Y$$

  This property is called the *triangle inequality*.

These properties turn out to capture the essential features of a distance function. Functions that satisfy them tend to meet people's expectations for how a distance function should behave. A function that satisfies these properties is called a *metric*.

1.4 Open balls

Given a point x, it's often useful to know which points are within a certain distance of x. We define the *open ball* of radius r around x as the set

$$B_x(r) = \{y \in Y : d(x,y) < r\}.$$

Here are some examples.

- [Sketch $B_x(r)$ in $\mathbb{R}$.]
- [Sketch $B_x(r)$ in $\mathbb{R}^2$.] The term "ball" comes from this example.
- [Sketch $B_x(r)$ in $\mathbb{T}$ for $r \leq \pi$ and $\pi \leq r$.]
- In $\mathbf{2}^{\mathbb{N}}$, the open ball $B_x(2^{-n})$ is the set of sequences which match x for the first $n+1$ digits.

1.5 Limits

Once we have a metric on Y, we can define the limits of sequences just like we do in $\mathbb{R}$.

Informal definition. The point $p \in Y$ is a *limit* of the sequence $y_1, y_2, y_3, y_4, \ldots$ if we can get y_n as close as we want to p just by making n large enough.

Formal definition. The point $p \in Y$ is a *limit* of $y_1, y_2, y_3, y_4, \ldots$ if for every radius $r > 0$, no matter how small, the sequence has a "tail" $y_N, y_{N+1}, y_{N+2}, y_{N+3}, \ldots$ that always stays inside $B_p(r)$.

When you define limits in terms of distance, like we'll always do in this course, a sequence can have at most one limit.[2] So, it makes sense to use the shorthand $\lim_{n\to\infty} y_n = p$ when we want to say p is a limit of $y_1, y_2, y_3, y_4, \ldots$.

2 Attraction and repulsion in a general state space

2.1 Definition

Consider a dynamical system with state space Y and dynamical map F. The point $p \in Y$ is a fixed point of F.

- A *basin of attraction* for p is an open ball U with the following properties.
  - U contains p.
  - Every orbit starting in U stays in U forever.
  - Every orbit starting in U limits to p.

  If there's a basin of attraction for p, we say p is *attracting*.[3]

  (The second property is equivalent to the property that $F(U) \subset U$, using our shorthand from last week.)

- A *region of repulsion* for p is an open ball U with the following properties.
  - U contains p.
  - Every orbit starting in U eventually leaves U, unless it starts at p.
    (It only has to leave once; it can come back later.)

  If there's a region of repulsion for p, we say p is *repelling*.[4]

[2] For the one or two students who've studied topology: this comes from the more general fact that a sequence in a Hausdorff space can have at most one limit. A topology defined in terms of a metric is always Hausdorff.

[3] The textbook uses the term *weakly attracting*.

[4] The textbook uses the term *weakly repelling*.

2.2 Examples

2.2.1 The doubling map

We learned earlier that the doubling map $D\colon \mathbb{T} \to \mathbb{T}$ has one fixed point: the angle 0. It turns out to be repelling. Here's how to see this.

- The easy way: use the linearization trick we learned last week, which works for the state space $\mathbb{T}$ as well as for $\mathbb{R}$. Observe that D is differentiable near 0, with $|D'(t)| > 1$. It follows that 0 is repelling.

- The straightforward way: find a region of repulsion. The ball $B_0(\frac{\pi}{2})$ will do the trick. Pick any point $\theta \in B_0(\frac{\pi}{2})$ other than 0, and express it as a number $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Observe that $D^n(t) \equiv 2^n t$. We can always find a whole number n with $|2^n t| \in [\frac{\pi}{2}, \pi)$.

2.2.2 The shift map

We learned earlier that the shift map $S\colon \mathbf{2}^{\mathbb{N}} \to \mathbf{2}^{\mathbb{N}}$ has two fixed points: $\overline{\mathtt{0}}$ and $\overline{\mathtt{1}}$. They both turn out to be repelling. Here's how to show this.

I claim that $B_{\overline{\mathtt{0}}}(1)$ is a region of repulsion for $\overline{\mathtt{0}}$. To see why, pick any point $x \in B_{\overline{\mathtt{0}}}(1)$ other than $\overline{\mathtt{0}}$. Since $1 = 2^{-0}$, the sequences in $B_{\overline{\mathtt{0}}}(1)$ are the ones that have the same first digit as $\overline{\mathtt{0}}$. Hence,

$$x = \mathtt{0}\,\square\square\square\square\square\,.$$

Since x is not $\overline{\mathtt{0}}$, it must have a $\mathtt{1}$ somewhere.

$$x = \underbrace{\mathtt{0}\,\square\square\square}_{n \text{ digits}}\,\mathtt{1}\,\square\square\,.$$

Let's say there's a $\mathtt{1}$ with n digits before it. Since S^n erases the first n digits, $S^n(x)$ starts with a $\mathtt{1}$.

$$S^n(x) = \mathtt{1}\,\square\square$$

Hence, $S^n(x)$ is not in $B_{\overline{\mathtt{0}}}(1)$.

In summary, we've shown that for any $x \in B_{\overline{\mathtt{0}}}(1)$ other than $\overline{\mathtt{0}}$, there's some time n at which $S^n(x)$ leaves $B_{\overline{\mathtt{0}}}(1)$. That means $B_{\overline{\mathtt{0}}}(1)$ is a region of repulsion.

2.2.3 Sweeping away the `1`s

In the homework, we defined a dynamical map $A\colon \mathbf{2}^{\mathbb{N}} \to \mathbf{2}^{\mathbb{N}}$.

- Defining rule: when you apply A, each `1` that's followed by a `0` turns into a `0`.

- Fixed points:

$$p_n = \underbrace{\mathtt{000\ldots0}}_{n}\,\mathtt{11111}\ldots$$

$$q = \mathtt{000000000000}\ldots$$

The fixed point p_n turns out to be repelling for every $n \in \{0, 1, 2 \ldots\}$. The open ball $B_{p_n}(2^{-n})$ is a region of repulsion. To see why, consider any $x \in B_{p_n}(2^{-n})$ other than p_n. By the definition of an open ball, $d(p_n, x) < 2^{-n}$, so x matches p_n for at least the first $n+1$ digits:

$$x = \underbrace{\texttt{000}\ldots\texttt{0}}_{n}\texttt{1}\colorbox{lightgray}{}.$$

Since $x \neq p_n$, there must be a `0` somewhere after that initial `1`:

$$x = \underbrace{\texttt{000}\ldots\texttt{0}}_{n}\texttt{1}\colorbox{lightgray}{}\texttt{0}\colorbox{lightgray}{}.$$

Hence, the orbit of x eventually reaches

$$\underbrace{\texttt{000}\ldots\texttt{0}}_{n}\texttt{0}\colorbox{lightgray}{},$$

which is outside $B_{p_n}(2^{-n})$.