Week 1 notes

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Chaos, fractals, and dynamics
MAT 335, Winter 2019

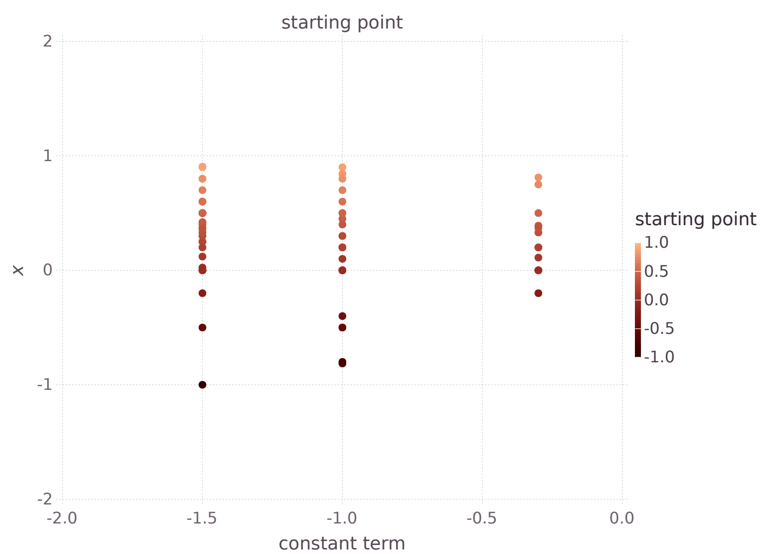
1 Opening activity

Take the function $F(x) = x^2 - 0.3$. It's a quadratic function, something you probably feel familiar with. Here's what I'd like to to do with it.

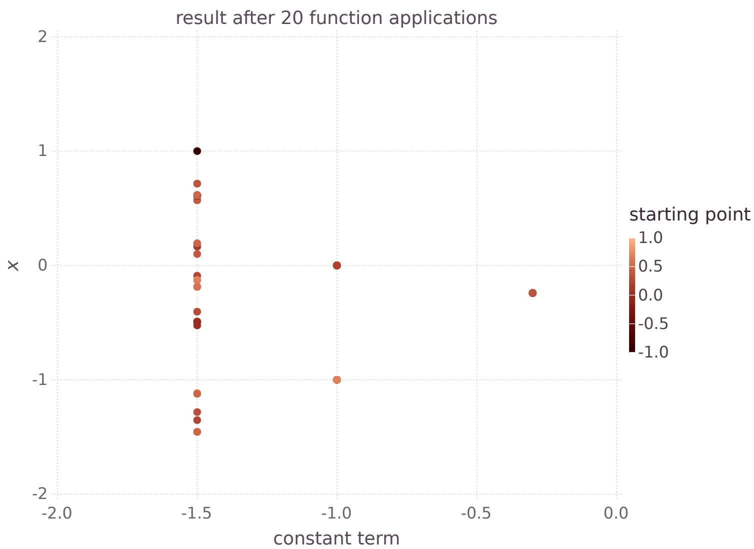
1. Pick a point $x \in (-1, 1)$.
2. Use a calculator to find $\underbrace{F(\ldots F(F(F}_{\text{20 times}}(x)))\ldots)$.

Do the same thing with the functions $G(x) = x^2 - 1$ and $H(x) = x^2 - 1.5$.

Here are the starting points you chose,

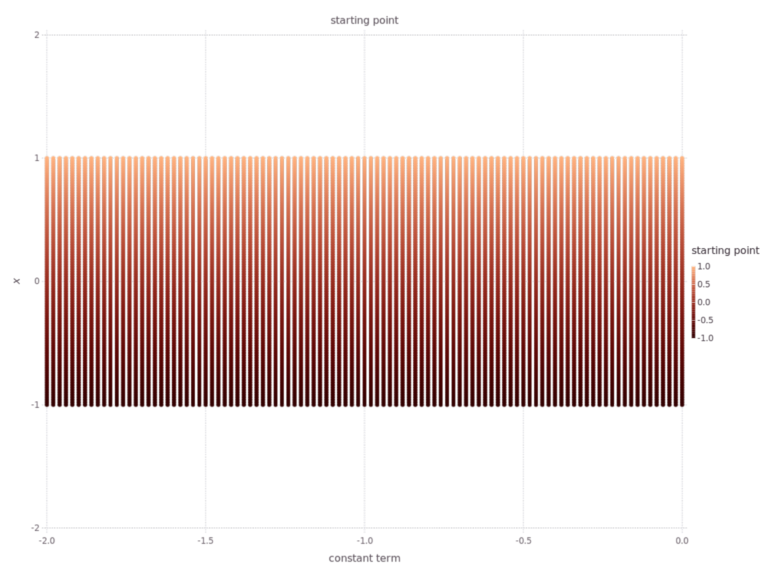


and here's where you ended up.

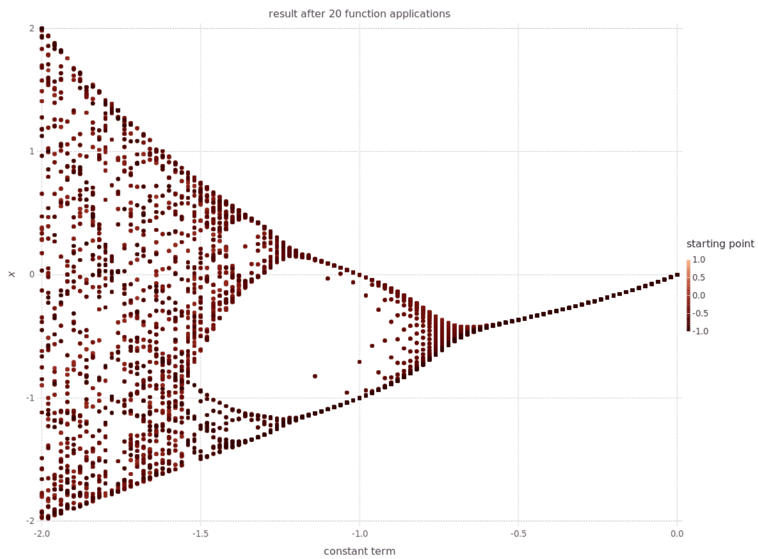


We started in lots of different places, but the people using F all ended up really close to the same point: $-0.24161\ldots$. Everyone using G ended up really close to either 0 or -1. In fact, as some of you noticed, you ended up bouncing back and forth between 0 and -1 every time you applied G. The people using H, on the other hand, seemed to end up in random places. The only hint of a pattern is what looks like a gap just above -1.

I used my computer to do the same experiment more systematically, with starting points covering the whole interval $(-1, 1)$ and functions ranging from $x^2 - 2$ to $x^2 - 0$. Here are my starting points,



and here's where I ended up.



Take a look at the slides to see what happened in between. This more detailed picture emphasizes the patterns we saw earlier for F and G, and confirms that the results for H really do have a gap just above -1.

The familiar, simple functions F, G, and H act in surprising and complicated ways when you apply them repeatedly. They also act very differently from one another, even though the look similar to begin with. Our goal in this course is to understand how that happens, both for these functions and for others.

2 Dynamical systems

2.1 Definition

The functions F, G, and H each describe a way of moving around on the real line. If you repeat the same movement over and over, where do you end up? *Dynamics* is the branch of math that tries to answer this question.

A repeated movement is called a *dynamical system*. It's typically described in two parts:

- The space you're moving in (the *state space*).
- How to move (the *dynamical map*).

2.2 Examples

2.2.1 (Standard) quadratic maps

State space: The real line. Shorthand: $\mathbb{R}$.

Dynamical map:[1] $Q_c(x) = x^2 + c$.

[1] Corrected. The sign of the constant term was wrong in the previous verison.

2.2.2 Rotation maps

State space: The unit circle. Shorthand: $\mathbb{T}$.

Dynamical map: Rotate the circle α radians counterclockwise. In symbols,[2] $R_\alpha(\theta) \equiv \theta + \alpha$.

2.2.3 The doubling map

State space: $\mathbb{T}$.

Dynamical map: $D(\theta) \equiv 2\theta$.

2.2.4 The shift map

State space: The set of sequences of 0s and 1s. Shorthand: $\mathbf{2}^{\mathbb{N}}$. Example points:

$$\begin{gathered}\texttt{00011111111}\ldots\\ \texttt{01011010010}\ldots.\end{gathered}$$

Dynamical map: Erase the first digit. Example:

$$\begin{aligned} w &= \texttt{00011111111}\ldots \\ S(w) &= \texttt{0011111111}\ldots. \end{aligned}$$

3 Two ways to look at repetition

Say we have a dynamical system with dynamical map F and state space Y. (In symbols, we'll write $F\colon Y \to Y$, as a shorthand for "F is a map from Y to Y.") Here are two complementary ways to look at what happens when you apply F repeatedly. We'll use both of them throughout the course.

3.1 Follow individual points

For each $y \in Y$, look at the sequence of points

$$\begin{gathered} y \\ F(y) \\ F(F(y)) \\ F(F(F(y))) \\ \vdots \end{gathered}$$

This is called the *orbit* of y.

[2] I'll use the symbol $\equiv$ for equivalence of angles, with $\theta \equiv \theta + 2\pi$. Since we usually express angles in terms of numbers, it's useful to have a special symbol to remind us what we're comparing.

3.2 Watch the whole state space

Look at the sequence of functions

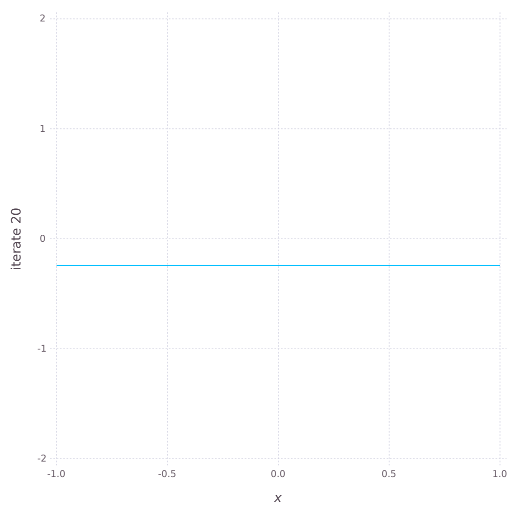
$$\begin{gathered} F \\ F \circ F \\ F \circ F \circ F \\ \vdots \end{gathered}$$

The function

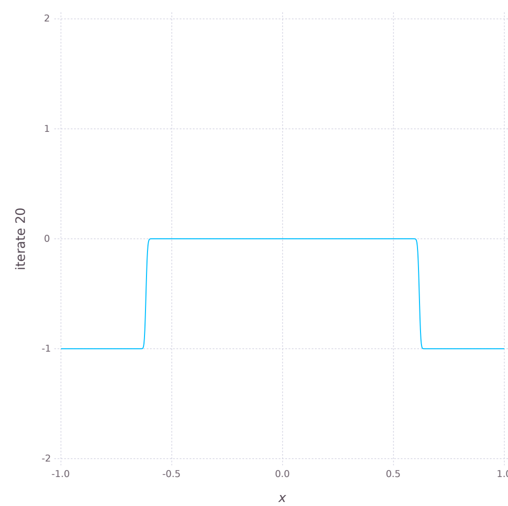
$$\underbrace{F \circ F \circ \ldots \circ F}_{n \text{ times}}$$

is called the nth *iterate* of F. We'll call it F^n for short.[3]

The nth iterate of F describes how the whole state space moves around after n applications of F. As an example, let's graph the 20th iterates of the quadratic maps from the opening activity, plus another standard quadratic map we haven't looked at before.

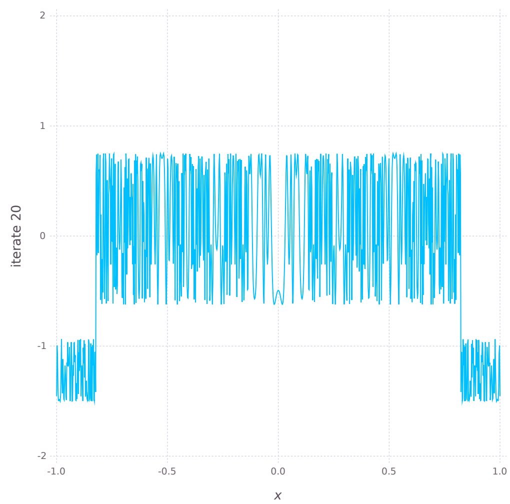


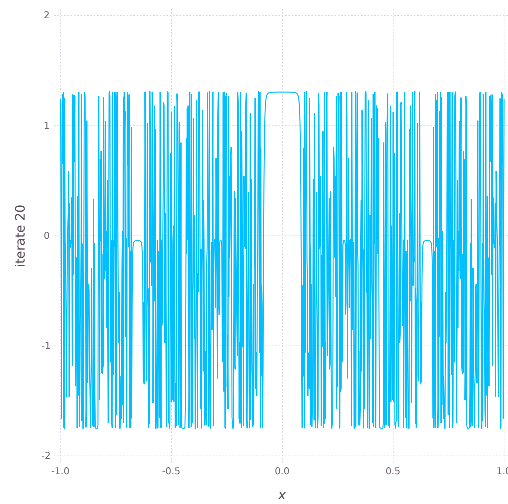
$Q_{-0.3}(x) = x^2 - 0.3$



$Q_{-1}(x) = x^2 - 1$

[3]There's an unfortunate conflict between this notation and an old-fashioned notation for raising the output of F to the nth power. In this course, $F^2(x)$ always means $F(F(x))$. If we ever want to talk about a power of the output of F, we'll do it in the modern way: $F(x)^2 = F(x)\,F(x)$.





$Q_{-1.5}(x) = x^2 - 1.5$ $\qquad$ $Q_{-1.75}(x) = x^2 - 1.75$

See the slides for graphs of all the iterates from 1 through 20. These graphs show the same patterns we saw in the activity, ando also reveal more.

The graph of ${Q_{-1}}^{20}$, for instance, shows which points end up at 0 and which end up at -1. It also reveals small intervals of points that don't end up near 0 or -1. We missed these when we were only following individual points, because a randomly chosen starting point is unlikely to fall in one of them.

In the graph of ${Q_{-1.5}}^{20}$, we can see that whether a point ends up above or below the gap depends in a simple way on where it starts, even though everything else looks random. This graph also reveals small intervals of points that end up in the gap.

4 Fixed points

4.1 Definition

Let's think back to our experiments with the quadratic map $F(x) = x^2 - 0.3$ on $\mathbb{R}$. The orbits of points in $(-1, 1)$ all seem to approach the same point, $m = -0.24161\ldots$. This point turns out to be a *fixed point* of F: it has the property that $F(m) = m$. Fixed points often make good landmarks in the state space of a dynamical system.

4.2 An application

In this particular case, knowing that m is a fixed point can help us find a formula for it. Here's how. The fixed points of F are the solutions of the equation

$$\begin{aligned} F(x) &= x \\ x^2 - 0.3 &= x \\ x^2 - x - 0.3 &= 0. \end{aligned}$$

There are two solutions, and thus two fixed points:

$$\begin{aligned} p_{\pm} &= \frac{1}{2}\left(1 \pm \sqrt{1 + 4 \cdot 0.3}\right) \\ &= 0.5 \pm \sqrt{0.55}. \end{aligned}$$

From the decimal expressions

$$\begin{aligned} p_{+} &= 1.24161\ldots \\ p_{-} &= -0.24161\ldots, \end{aligned}$$

we can see that $m = p_{-}$.

4.3 Fixed points of other dynamical systems

4.3.1 Rotation map

Take a rotation map $R_\alpha \colon \mathbb{T} \to \mathbb{T}$, with $\alpha \not\equiv 0$. Looking at how points move around the circle when you rotate, you should be able to see pretty easily that θ and $R_\alpha(\theta)$ are never the same, so R_α has no fixed points.

4.3.2 The doubling map

The fixed points of the doubling map $D \colon \mathbb{T} \to \mathbb{T}$ are the solutions of the equation

$$\begin{aligned} D(\theta) &\equiv \theta \\ 2\theta &\equiv \theta \\ 2\theta - \theta &\equiv 0 \\ \theta &\equiv 0, \end{aligned}$$

so the only fixed point is the angle 0.

4.3.3 The shift map

To be a fixed point of the shift map $S \colon \mathbf{2}^{\mathbb{N}} \to \mathbf{2}^{\mathbb{N}}$, a sequence has to have the property that every digit is the same as the next digit. On the other hand, every sequence with this property is a fixed point. Hence, S has two fixed points: `00000000000`... and `11111111111`....

5 Eventually fixed points

5.1 Example

Look at the orbit of $\frac{2\pi}{8}$ under the doubling map $D\colon \mathbb{T} \to \mathbb{T}$.

n	0	1	2	3	4	5	…
$F^n(\frac{2\pi}{8})$	$\frac{2\pi}{8}$	$\frac{2\pi}{4}$	$\frac{2\pi}{2}$	2π	2π	2π	…

Although it's not fixed, it eventually reaches the fixed point 2π. After that, of course, it stays where it is. This suggests a generalization of the idea of a fixed point.

5.2 Definition

An *eventually fixed point* of a dynamical map $F\colon Y \to Y$ is a point whose orbit eventually reaches a fixed point. Equivalently, it's a point $y \in Y$ with the property that $F^{n+1}(y) = F^n(y)$ for some n.

5.3 Another example

A sequence that ends in an infinite string of `0`s or an infinite string of `1`s is an eventually fixed point of the shift map $S\colon \mathbf{2}^{\mathbb{N}} \to \mathbf{2}^{\mathbb{N}}$. Every eventually fixed point of S looks like this.

6 Periodic points

The quadratic map $Q_{-1}(x) = x^2 - 1$ exchanges 0 and -1. As a result, the orbits of 0 and -1 both return to their starting points after two steps. This suggests another generalization of the idea of a fixed point.

A *periodic point* of a dynamical map $F\colon Y \to Y$ is a point whose orbit eventually returns to its starting point.

A periodic point y of F is called *n-periodic* if $F^n(y) = y$. Once a periodic orbit gets back to its starting point, it can go around again, so every n-periodic point is also $2n$-periodic, $3n$-periodic, and so on.

It's good to know how many steps a periodic point will take to return to itself for the first time. I'll call this number the point's *minimum period*. (The textbook uses the term *prime period*, which I'll avoid, because I find it confusing.)