Term Test 1

February 12
7:10 – 9:00 p.m.

Chaos, fractals, and dynamics
MAT 335, Winter 2019

Instructions

- There are five problems on the test.
- Show your calculations, and explain your reasoning. Your goal is for the graders to understand how you got your answers, and to be convinced your reasoning makes sense.
- Do not discuss the test until 6 p.m. Wednesday, since there will be an alternate sitting.
- Do not turn the page until you're told to start the test.
- Before the test starts, write your name and student number above.
- Only the work written in this booklet will be marked. Don't write near the staple.
- If your work on a problem continues into the extra space at the end of the booklet, say so in the space for the problem.
- If you need scrap paper for scratch work, raise your hand.
- If you have a question or need to use the washroom, raise your hand.
- You may use a ruler for graphical analysis. No other computing devices are allowed.

1. For the shift map $S\colon \mathbf{2}^{\mathbb{N}} \to \mathbf{2}^{\mathbb{N}}$, list the periodic points...

   (a) ...with minimum period 1.

   (b) ...with minimum period 2.

   (c) ...with minimum period 4.

   To save time, you only have to list one point from each orbit.

   Your answer should include a brief explanation of how you found the periodic points, and how you know that you found them all.

   Marking

   - 6 points for having correct answers.
   - 3 points for having a decent explanation.

   The explanation can be very brief, and it doesn't have to be airtight.

   Sample solution

   (a) The 1-periodic points of the shift map are the sequences in which each digit is equal to the next digit. Hence, a 1-periodic sequence is determined by its first digit, and any choice of first digit is possible. The 1-periodic points are therefore

   $$\overline{\mathtt{0}}, \quad \overline{\mathtt{1}}.$$

   A 1-periodic point automatically has minimum period 1.

   (b) The 2-periodic points of the shift map are the sequences in which each digit is equal to the digit two places to the right. Hence, a 2-periodic sequence is determined by its first two digits, and any choice of the first two digits is possible. Writing down the four possible choices of first two digits, and excluding the two that already appeared in our list of 1-periodic sequences, we see that

   $$\overline{\mathtt{01}}, \quad \overline{\mathtt{10}}$$

   are the sequences of minimum period 2.

   (c) By the same reasoning we used in the previous parts, a 4-periodic sequence is determined by its first four digits, and any choice of the first three digits is possible. Writing down the sixteen possible choices of first four digits, and excluding the four that already appeared in our lists of 1-periodic and 2-periodic sequences, we see that the sequences of minimum period 4 are

   $$\overline{\mathtt{1000}}, \quad \overline{\mathtt{1100}}, \quad \overline{\mathtt{1110}}$$

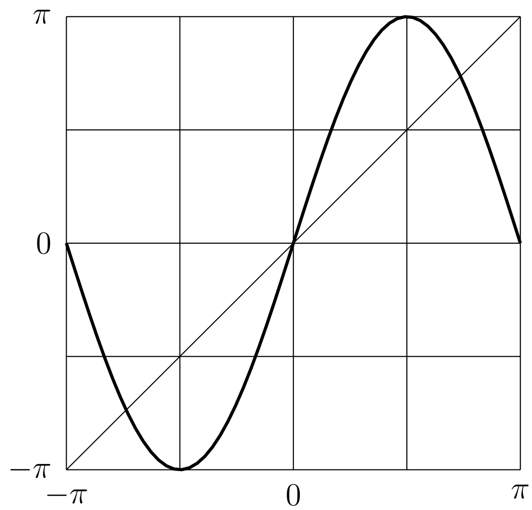
   and the other points in their orbits.

Continue problem 1.

2. Consider the dynamical system with state space $\mathbb{T}$ and dynamical map

$$G(x) = \pi \sin(x).$$

Here's the graph of G.



- Find every point whose orbit reaches $0 \in \mathbb{T}$ within three steps.

Marking

- 6 points for the whole problem.

Sample solution

The point 0 reaches 0 in zero steps.

Each solution of $\pi \sin(x) \equiv 0$ reaches 0 within one step. The solutions are 0 and π.

Each solution of $\pi \sin(x) \equiv \pi$ reaches 0 within two steps. The solutions are $\frac{\pi}{2}$ and $-\frac{\pi}{2}$.

Each solution of $\pi \sin(x) = \frac{\pi}{2}$ and each solution of $\pi \sin(x) \equiv -\frac{\pi}{2}$ reaches 0 within three steps. The solutions of the first equation are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$. The solutions of the second equation are $-\frac{\pi}{6}$ and $-\frac{5\pi}{6}$.

Continue problem 2.

3. Consider the following dynamical system.

   State space: $\mathbf{2}^{\mathbb{N}}$.

   Dynamical map: Each `1` that's followed by a `0` turns into a `0`.

   Let's call this map A. As a demonstration, here's what A does to one point in $\mathbf{2}^{\mathbb{N}}$.

$$\begin{aligned} w &= \texttt{0011100110111110100101110}\ldots \\ A(w) &= \texttt{0011}\underline{\texttt{0}}\texttt{001}\underline{\texttt{0}}\texttt{01111}\underline{\texttt{0}}\texttt{0}\underline{\texttt{0}}\texttt{00}\underline{\texttt{0}}\texttt{011}\underline{\texttt{0}}\texttt{0}\ldots \end{aligned}$$

   The changed digits are underlined.

   (a) Describe all the fixed points of A.

   (b) Does A have any periodic points other than the fixed points? If it does, find one. If it doesn't, explain how you know it doesn't.

Marking

- 3 points for part a.
- 6 points for part b.

In part b, for full credit, the explanation has to include two key ideas:

- If $w \in \mathbf{2}^{\mathbb{N}}$ is not a fixed point, then applying A to w turns a `1` into a `0`.
- Once A has turned a `1` into a `0`, there's no way for it to turn that `0` back into a `1`.

Sample solution

(a) The map A changes every `1` which is followed by a `0`, and doesn't change anything else. Hence, the fixed points of A are the sequences in which `10` never occurs. Using the bar notation from homework 1, these sequences are

$$\overline{\texttt{1}}, \quad \texttt{0}\overline{\texttt{1}}, \quad \texttt{00}\overline{\texttt{1}}, \quad \texttt{000}\overline{\texttt{1}}, \quad \texttt{0000}\overline{\texttt{1}}, \quad \ldots$$

and also

$$\overline{\texttt{0}}.$$

(b) Every periodic point of A is a fixed point of A. This is because the only way A can change a sequence is by changing `0`s into `1`s. This has the following consequences.

- If $A(w) \neq w$, then w has a `0` at some position where w has a `1`.
- If $v \in \mathbf{2}^{\mathbb{N}}$ has a `0` at some position where $w \in \mathbf{2}^{\mathbb{N}}$ has a `1`, then the orbit of v can't include w.

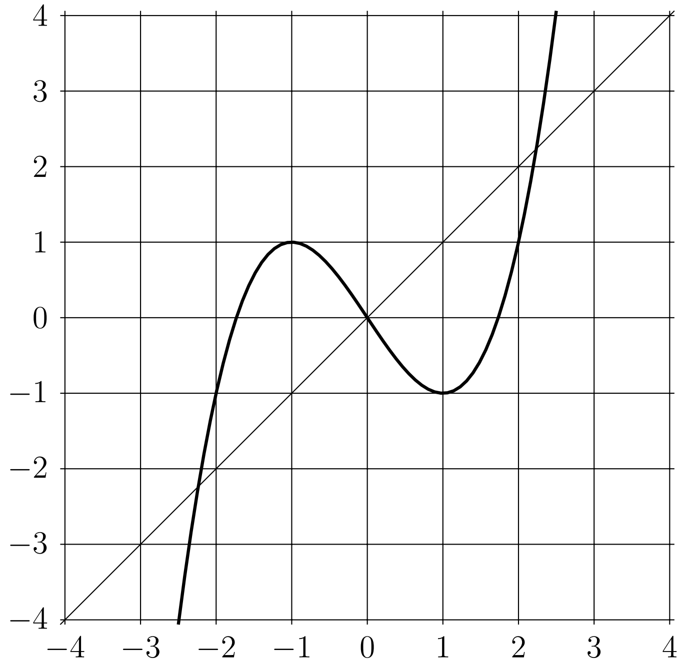
Putting these facts together, we see that if $A(w) \neq w$, then the orbit of $A(w)$ can't include w. In other words, if $w \in \mathbf{2}^{\mathbb{N}}$ isn t fixed, then w can't be periodic.

Continue problem 3.

4. Consider the dynamical system with state space $\mathbb{R}$ and dynamical map

$$F(x) = \tfrac{1}{2}(x^3 - 3x).$$

Here's the graph of F.



(a) List all the fixed points of F.

(b) Classify each fixed point as attracting, repelling, or neither.

(c) List all the periodic points of F.

(d) Describe all the points whose orbits eventually enter the interval $[0, 1]$.

Marking

- 3 points for part a.
- 3 points for part b.
- 3 points for part c.
- 3 points for part d.

Sample solution

(a) The fixed points of F are the solutions of

$$\begin{aligned} F(x) &= x \\ \tfrac{1}{2}(x^3 - 3x) &= x \\ x^3 &= 5x. \end{aligned}$$

This equation is satisfied if and only if $x = 0$ or $x^2 = 5$. The fixed points of F are therefore $-\sqrt{5}$, 0, $\sqrt{5}$.

(b) Let's try the derivative test. (The function F is differentiable everywhere, so the conditions of the test are satisfied.) Using the formula $F'(x) = \frac{3}{2}(x^2 - 1)$, we calculate

$$\begin{aligned} F'(-\sqrt{5}) &= 6 \\ F'(0) &= -\tfrac{3}{2} \\ F'(\sqrt{5}) &= 6, \end{aligned}$$

and learn that $|F'| > 1$ at all the fixed points. Hence, all the fixed points are repelling.

(c) Since $F(1) = -1$ and $F(-1) = 1$, the points 1 and -1 are 2-periodic. The fixed points $-\sqrt{5}$, 0, $\sqrt{5}$ are 1-periodic.

The map F has no other periodic points. To see why, first observe that the graph of F lies below the diagonal on the interval $(-\infty, -\sqrt{5})$, and above the diagonal on the interval $(\sqrt{5}, \infty)$. Hence, orbits starting in $(-\infty, -\sqrt{5})$ move left at every step, while those starting in $(\sqrt{5}, \infty)$ move right at every step. That means an orbit starting in $(-\infty, -\sqrt{5})$ or $(\sqrt{5}, \infty)$ can never return to its starting point.

Orbits starting in $(-\sqrt{5}, -1)$ and $(1, \sqrt{5})$ eventually enter $(-1, 1)$, and points in $(-1, 1)$ stay in that interval. Hence, orbits starting in $(-\sqrt{5}, -1)$ and $(1, \sqrt{5})$ also can never return to their starting points.

Now, compare the graph of F to the line of slope -1 through the point $(0, 0)$. The graph of F lies above this line on the interval $(-1, 0)$, and below this line on the interval $(0, 1)$. As a result, an orbit starting in $(-1, 0)$ and $(0, 1)$ is always moving further from 0, so it can't return to its starting point.

We now see that an orbit starting at a point other than $-\sqrt{5}, -1, 0, 1, \sqrt{5}$ can never return to its starting point. Hence, no points other than the ones just listed can be periodic.

(d) As we saw in part c, orbits starting in $(-\infty, -\sqrt{5})$ fly off to the left, and orbits starting in $(\sqrt{5}, \infty)$ fly off to the right. These orbits can never enter $[0, 1]$. Since $-\sqrt{5}$ and $\sqrt{5}$ are fixed points, their orbits can't enter $[0, 1]$ either.

Combining a few conclusions from part c, we can deduce that orbits starting in $(-\sqrt{5}, \sqrt{5})$ eventually end up in $[-1, 1]$. Furthermore, we can see from the graph that F sends points in $[-1, 0]$ into $[0, 1]$. Hence, every orbit starting in $(-\sqrt{5}, \sqrt{5})$ eventually enters $[0, 1]$.

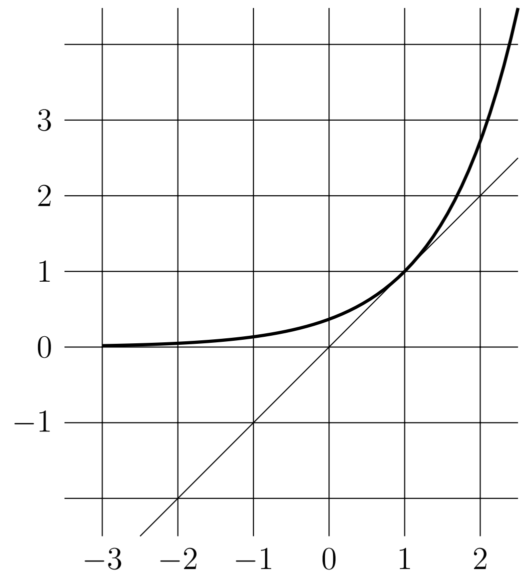
In summary, the points whose orbits eventually enter $[0, 1]$ are the points in the interval $(-\sqrt{5}, \sqrt{5})$.

Continue problem 4.

5. Consider the dynamical system with state space $\mathbb{R}$ and dynamical map

$$H(x) = e^{x-1}.$$

Here's the graph of H.



The point 1 is a fixed point of H.

- Is the interval $(0, 2)$ a basin of attraction for 1? Use graphical or algebraic reasoning to justify your answer.

Marking

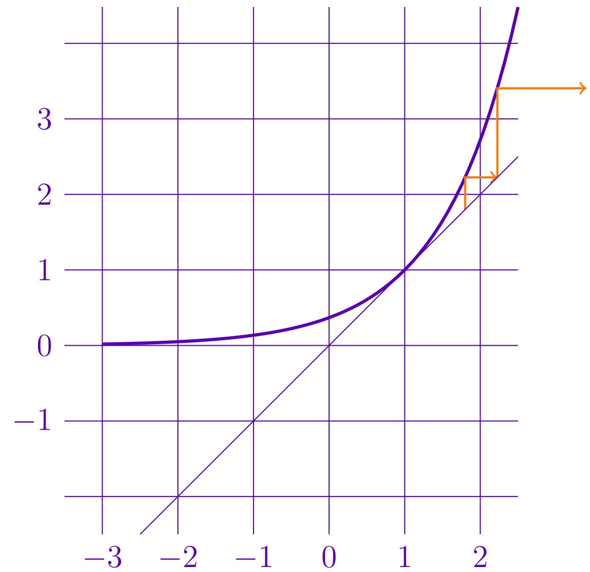
- 6 points for the whole problem.

A basin of attraction is an open ball with three special properties (see Section 2.1 of the week 3 notes.) For full credit, a solution must do the following things.

- Recognize that $(0, 2)$ is not a basin of attraction for 1.
-

There are many possible solution methods. Some of my favorites are shown below.

A graphical solution



The interval $(0,2)$ is not a basin of attraction for 1, because not every orbit starting in $(0,2)$ stays in $(0,2)$ forever. As an example, consider the orbit sketched above. The orbit starts at a place where the graph of h sits above 2, so it leaves $(0,2)$ after one step.

An algebraic solution

The interval $(0,2)$ is not a basin of attraction for 1, because not every orbit starting in $(0,2)$ stays in $(0,2)$ forever. As an example, consider the orbit starting at 1.9. Since

$$\begin{aligned} H(1.9) &= e^{0.9} \\ &\geq 2, \end{aligned}$$

this orbit eventually leaves $(0,2)$.

An extra thorough algebraic solution

The interval $(0,2)$ is not a basin of attraction for 1, because not every orbit starting in $(0,2)$ stays in $(0,2)$ forever. As an example, consider the orbit starting at $1+\ln(2)$. As we'll verify below, $1+\ln(2)$ is in $(0,2)$. However,

$$\begin{aligned} H(1+\ln(2)) &= e^{\ln(2)} \\ &= 2, \end{aligned}$$

which is outside $(0,2)$.

To check that $1+\ln(2)$ is in $(0,2)$, recall that the natural logarithm function is increasing: $a < b$ implies $\ln(a) < \ln(b)$. We can thus reason as follows:

$$\begin{aligned} 1 &< 2 < e \\ \ln(1) &< \ln(2) < \ln(e) \\ 0 &< \ln(2) < 1 \\ 1 &< 1+\ln(2) < 2. \end{aligned}$$

Continue problem 5.

Extra space

Extra space

Extra space