Homework 3

Due on Crowdmark
March 4, 11 a.m.

Chaos, fractals, and dynamics
MAT 335, Winter 2019

Show your calculations, and explain your reasoning. Your goal is for the graders to understand how you got your answers, and to be convinced that your reasoning makes sense.

1 Standardizing quadratic maps

In week 1, when we first met the dynamical maps $Q_u(x) = x^2 + u$ on the state space $\mathbb{R}$, I introduced them as the "standard quadratic maps." Now that we've learned about semiconjugacy, I can explain why I chose that name.

a. Find a semiconjugacy from the quadratic map $F(x) = x^2 + 6x + 5$ to the standard quadratic map Q_{-1}. *(Corrected: the previous version had* 3 *as the coefficient of* x*.)*

   HINT: Look for constants a, b that make $\psi(x) = ax+b$ a semiconjugacy from F to Q_{-1}. You should make sure you found a semiconjugacy by checking the four properties a semiconjugacy needs to have. You can take it as given that $\psi(x) = ax+b$ is continuous for any choice of a and b, but you should think about why this is true.

b. Find a semiconjugacy from the quadratic map $G(x) = 2x^2 - 3$ to a standard quadratic map Q_u.

   HINT: Look for constants a, b, u that make $\psi(x) = ax + b$ a semiconjugacy from G to Q_u.

c. Show that every quadratic map $P(x) = Ax^2+2Bx+C$, with $A \neq 0$, is semiconjugate to a standard quadratic map Q_u. Write formulas for the semiconjugacy and the constant u in terms of A, B, C.

Each of the semiconjugacies you'll find in this problem has a special property: it's invertible, and its inverse is a semiconjugacy too. A semiconjugacy that goes both ways like this is called a *conjugacy*. If two dynamical systems are connected by a conjugacy, they're the same for all practical purposes. So, this problem demonstrates that every quadratic map is the same, for all practical purposes, as one of our standard quadratic maps.

2 An itinerary function

Let's revisit a dynamical system we mentioned briefly in the first week of the course: the rotation map $R_2 \colon \mathbb{T} \to \mathbb{T}$, defined by the formula $R_2(\theta) \equiv \theta + 2$. The points in $\mathbb{T}$ whose orbits never hit 0 or π form a subset $\Lambda \subset \mathbb{T}$. Let's define a function $\tau \colon \Lambda \to \mathbf{2}^{\mathbb{N}}$ in the following way.

$$\text{the } n\text{th digit of } \tau(\theta) \text{ is } \begin{cases} 0 & \text{if } R_2^n(\theta) \in (0, \pi) \\ 1 & \text{if } R_2^n(\theta) \in (\pi, 2\pi) \end{cases}$$

Let's call the starting digit of a sequence the 0th digit.

You can find the sequence $\tau(\theta)$ by writing down the orbit of θ and then noting whether each point on the orbit is in the top half or the bottom half of the unit circle. For example, here's a calculation of the first five digits of $\tau(1)$.

n	0	1	2	3	4
$R_2^n(1)$	1.000...	3.000...	5.000...	0.716...	2.716...
nth digit of $\tau(1)$	0	0	1	0	0

Intuitively, the sequence $\tau(\theta)$ tells you when the orbit of θ visits the top and bottom halves of the unit circle. A function that gives this kind of information is called an *itinerary function*. Itinerary functions are helpful for understanding the orbits of many dynamical systems, including quadratic maps.

Surprisingly, the function τ is continuous! This problem will walk you through an argument that τ is continuous at 1. To understand what you're doing, you should review Section 2 of the week 5 notes.

a. Use a calculator to find the first twenty digits of $\tau(1)$. (The table above gives you the first five digits for free.)

b. Find a radius $r \in (0, \infty)$ small enough that τ sends every point in $B_1(r)$ into the target ball $B_{\tau(1)}(2^0)$. We'll use the shorthand $\tau(B_1(r)) \subset B_{\tau(1)}(2^0)$ to express this condition.

c. Find a radius $r \in (0, \infty)$ small enough that $\tau(B_1(r)) \subset B_{\tau(1)}(2^{-1})$.

d. Find a radius $r \in (0, \infty)$ small enough that $\tau(B_1(r)) \subset B_{\tau(1)}(2^{-2})$.

e. Convince me that if I gave you a whole number $k \geq 0$, you could find a radius $r \in (0, \infty)$ small enough that $\tau(B_1(r)) \subset B_{\tau(1)}(2^{-k})$. A good way to do this is to describe a step-by-step procedure you would use to find a value of r that works.

In each of the parts b, c, and d, you need to convince me that your value of r works. This can be done with a short calculation.