Homework 2

Due on Crowdmark
January 31, 11 a.m.

Chaos, fractals, and dynamics
MAT 335, Winter 2019

Show your calculations, and explain your reasoning. Your goal is for the graders to understand how you got your answers, and to be convinced that your reasoning makes sense.

In problems 1 – 3, describe the orbits of the given dynamical system using graphical analysis, and any other tools you'd like. Your description should include:

- Lists of all fixed points and periodic orbits.
- Classification of each fixed point and periodic orbit as attracting, repelling, or neither.
- A description of the long-term behavior of every orbit.

1 Graphical analysis of a quadratic map

The map $F(x) = 1 - x^2$ on the state space $\mathbb{R}$.

2 Graphical analysis of a map on the circle

The map $G(\theta) = \theta + \frac{\pi}{4}(1 + \cos\theta)$ on the state space $\mathbb{T}$.

HINT: you can graph G on any $[\alpha, \alpha + 2\pi]$ by $[\alpha, \alpha + 2\pi]$ square. Choose a starting angle α that makes the graph easy to read.

3 Graphical analysis of a rational map

The map

$$H(x) = \frac{1 - x}{1 + x}$$

on the state space "$\mathbb{R}$ with the point -1 removed." You can refer to the state space in symbols as $\mathbb{R} \setminus \{-1\}$. *(Corrected. The previous version gave $\mathbb{R}$ as the state space.)*

HINT: Sketch a cobweb plot to get a rough idea of what's going on, but then verify everything algebraically.

4 Sweeping the other way

Consider the following dynamical system.

State space: $\mathbf{2}^{\mathbb{N}}$.

Dynamical map: Each `0` that comes after a `1` turns into a `1`.

Let's call this map B. As a demonstration, here's what B does to one point in $\mathbf{2}^{\mathbb{N}}$.

$$w = \mathtt{00110001101110001001011}\ldots$$
$$B(w) = \mathtt{0011\underline{1}0011\underline{1}111\underline{1}001\underline{1}01\underline{1}11}\ldots$$

The changed digits are underlined.

a. Describe all the fixed points of B.

b. Classify each fixed point as attracting, repelling, or neither.

5 Stripes

Consider the following dynamical system.

> **State space:** $\mathbf{2}^{\mathbb{N}}$.
>
> **Dynamical map:** Each `0` that's followed by a `0` turns into a `1`, and each `1` that's followed by a `1` turns into a `0`.

Let's call this map E. As a demonstration, here's what E does to one point in $\mathbf{2}^{\mathbb{N}}$.

$$\begin{aligned} w &= \texttt{0011100110111110000101110}\ldots \\ E(w) &= \texttt{\underline{1}0\underline{00}1\underline{0}0\underline{0}10\underline{000}1\underline{111}010\underline{00}10}\ldots \end{aligned}$$

a. Find two fixed points of E, and convince the grader they're the only two. *(Corrected. The previous version claimed, incorrectly, that there was only one fixed point.)*

b. Find two points with minimum period 2, and convince the grader they're the only ones.

   HINT: You already know what E does to the first digit of each 2-digit block. Figure out what E^2 does to the first digit of each 3-digit block.

c. The two points with minimum period 2 form a 2-periodic orbit. Convince the grader that this orbit is repelling.

   HINT: To show that a 2-periodic orbit is repelling, you pick a point p on the orbit and show that it's a repelling fixed point of E^2. As a first step, consider a point w which first differs from p at the $(n+1)$st digit, and see what E^2 does to the first n digits of w.