Homework 2

Due on Crowdmark
January 31, 11 a.m.

Chaos, fractals, and dynamics
MAT 335, Winter 2019

Show your calculations, and explain your reasoning. Your goal is for the graders to understand how you got your answers, and to be convinced that your reasoning makes sense.

Marking guide

For each problem, in the "Solution" heading, I describe how to split the problem into gradable pieces, and how many points each piece is worth. Give the work for each gradable piece 0/3*,* 1/3*,* 2/3*, or* 3/3 *of the points available, according to the guidelines below. Record your mark for each piece on the page; I recommend using the built in ✘ and ✔ symbols in Crowdmark as shown.*

✘ **0/3** Does not show much understanding of what's going on.

✔ **1/3** Shows a basic understanding of what's going on, but doesn't get very far down any promising path to a solution.

✔✔ **2/3** Gets pretty far down a promising path to a solution, but doesn't get all the way there, due to a conceptual error, major computation errors, or significant omissions.

✔✔✔ **3/3** Gets basically all the way to a solution, with only superficial errors or omissions.

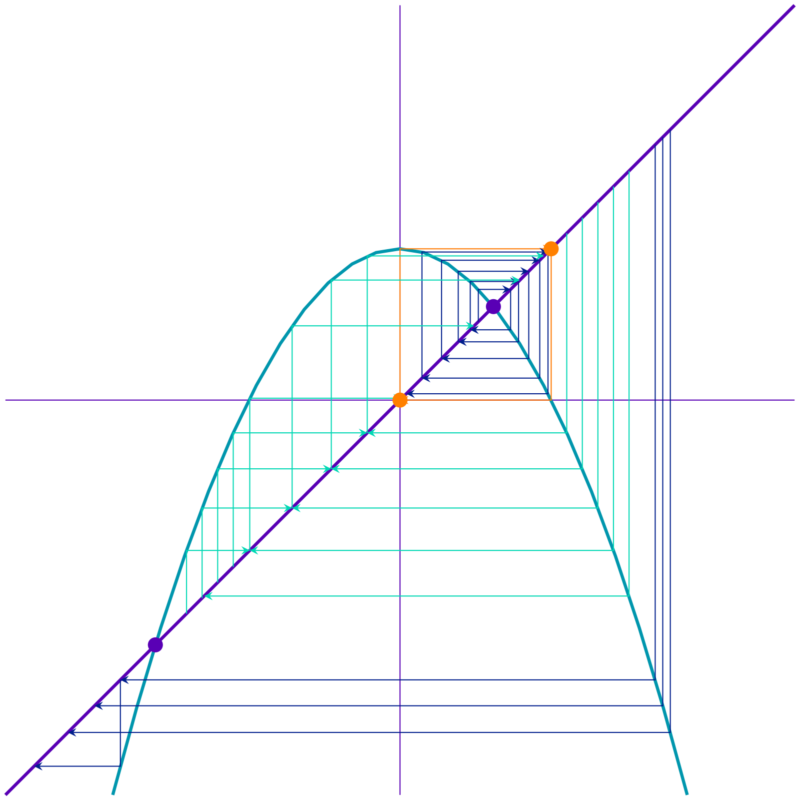
In problems 1 – 3, describe the orbits of the given dynamical system using graphical analysis, and any other tools you'd like. Your description should include:

- Lists of all fixed points and periodic orbits.
- Classification of each fixed point and periodic orbit as attracting, repelling, or neither.
- A description of the long-term behavior of every orbit.

1 Graphical analysis of a quadratic map

The map $F(x) = 1 - x^2$ on the state space $\mathbb{R}$.

Solution



The fixed points of F are the solutions of the equation

$$\begin{aligned} F(x) &= x \\ 1 - x^2 &= x \\ 0 &= x^2 + x - 1. \end{aligned}$$

Hence, the fixed points are $p_{\pm} = \frac{1}{2}\left(-1 \pm \sqrt{5}\right)$. Observe that $F'(p_{\pm}) = -1 \pm \sqrt{5}$, so $|F'|$ is greater than one at both fixed points. Hence, both fixed points are repelling. *[To use the derivative test, you technically need to verify that F is continuously differentiable around each fixed point, but it's okay if students skip that.]*

It's apparent from the graph, and can be verified algebraically, that 0 and 1 form a 2-periodic orbit. Observe that

$$\begin{aligned} F^2(x) &= 1 - (1 - x^2)^2 \\ &= 1 - (1 - 2x^2 + x^4) \\ &= x^2(2 - x^2), \end{aligned}$$

so $(F^2)'(0) = 0$. Hence, the orbit of 0 and 1 is attracting.

Points that start in the interval $(0, 1)$ stay there forever. Looking more closely at the graph, and knowing that p_+ is repelling, it's pretty clear that every orbit starting in $(0, 1) \setminus \{p_+\}$ approaches the orbit of 0 and 1.

Points in $(p_-, 0)$ hop to the right until they fall into the interval $(0, 1)$. From there, they either hit p_+ or approach the orbit of 0 and 1.

Points in $(1, -p_-)$ fall into $(p_-, 0)$. From there, they either hit p_+ or approach the orbit of 0 and 1.

Points in $(-\infty, p_-)$ hop to the left forever. Points in $(-p_-, \infty)$ fall into $(-\infty, p_-)$ and then hop to the left forever.

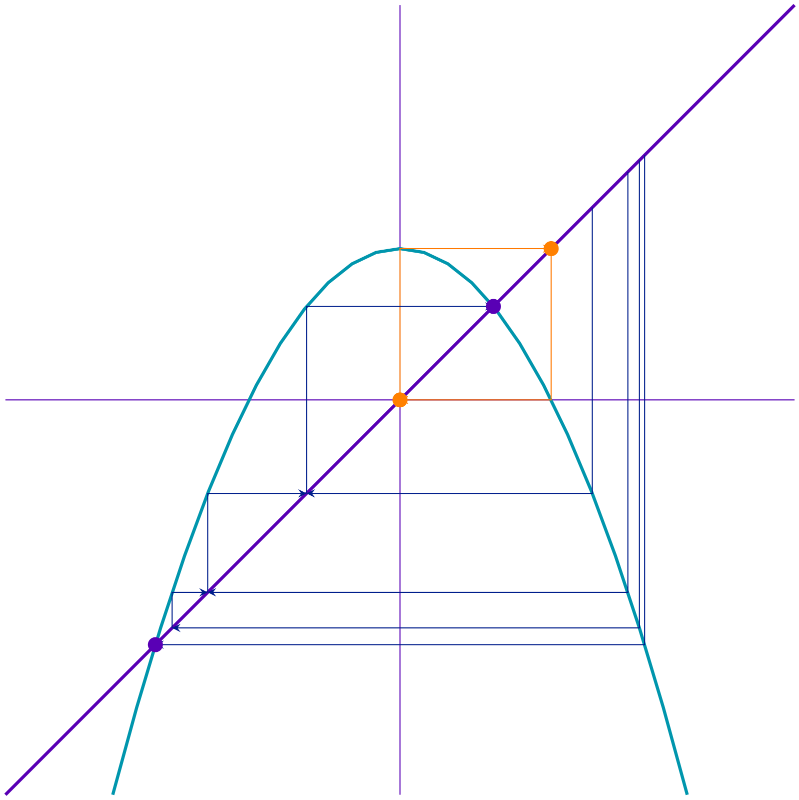
Here's an explicit description of the eventually fixed points *[not required for full credit]*. The point $-p_-$ eventually hits p_-, and it's the only point that does. The points

$$\begin{aligned}
m_1 &= -\sqrt{1 - p_+} \\
m_2 &= -\sqrt{1 - m_1} = -\sqrt{1 + \sqrt{1 - p_+}} \\
m_3 &= -\sqrt{1 - m_2} = -\sqrt{1 + \sqrt{1 + \sqrt{1 - p_+}}} \\
&\vdots
\end{aligned}$$

eventually hit p_+, and the points

$$\begin{aligned}
\ell_1 &= \sqrt{1 - m_1} = \sqrt{1 + \sqrt{1 - p_+}} \\
\ell_2 &= \sqrt{1 - m_2} = \sqrt{1 + \sqrt{1 + \sqrt{1 - p_+}}} \\
\ell_3 &= \sqrt{1 - m_3} = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 - p_+}}}} \\
&\vdots
\end{aligned}$$

do the same. The first few points on the list are sketched below.



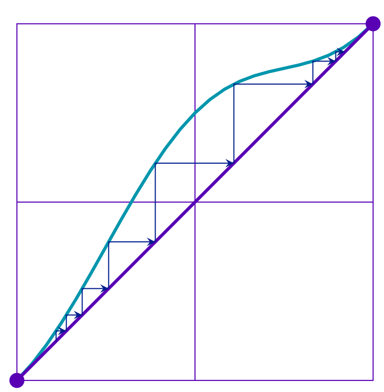
2 Graphical analysis of a map on the circle

The map $G(\theta) = \theta + \frac{\pi}{4}(1 + \cos\theta)$ on the state space $\mathbb{T}$.

HINT: you can graph G on any $[\alpha, \alpha + 2\pi]$ by $[\alpha, \alpha + 2\pi]$ square. Choose a starting angle α that makes the graph easy to read.

Solution

You can see algebraically that the only fixed point is π. Let's graph G on a $[-\pi, \pi]$ by $[-\pi, \pi]$ square.



The graph lies entirely above the diagonal, meeting the diagonal at the fixed point π. As a result, every orbit other than the one beginning at π steps steadily to the right, approaching π. (This implies that there are no fixed or periodic points other than π.)

It follows that π is attracting, with the whole circle as a basin of attraction. (The whole circle can be expressed as, say, the open ball around π with radius $\pi + \frac{1}{100}$.)

3 Graphical analysis of a rational map

The map

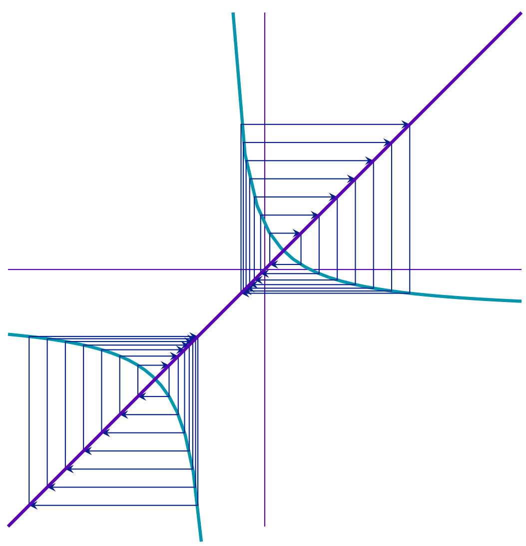
$$H(x) = \frac{1-x}{1+x}$$

on the state space "$\mathbb{R}$ with the point -1 removed." You can refer to the state space in symbols as $\mathbb{R} \setminus \{-1\}$. *(Corrected. The previous version gave $\mathbb{R}$ as the state space.)*

HINT: Sketch a cobweb plot to get a rough idea of what's going on, but then verify everything algebraically.

Solution

(The original version of the problem incorrectly gave $\mathbb{R}$ as the state space. Do not penalize students who say the state space is $\mathbb{R}$.)



The graph appears to by symmetric across the diagonal. This would imply that every point returns to its original position when you H twice. To verify, let's find a formula for H^2:

$$\begin{aligned}
H^2(x) &= \frac{1 - H(x)}{1 + H(x)} \\
&= \frac{1 - \frac{1-x}{1+x}}{1 + \frac{1-x}{1+x}} \\
&= \frac{(1+x) - (1-x)}{(1+x) + (1-x)} \\
&= \frac{2x}{2} \\
&= x.
\end{aligned}$$

We've now confirmed that $H^2(x) = x$, so every point in $\mathbb{R} \smallsetminus \{-1\}$ is 2-periodic for H.

We can see in the graph that there are two fixed points: one to the right of -1 and one to the left. They're the solutions of the equation

$$\begin{aligned}
H(x) &= x \\
\frac{1-x}{1+x} &= x \\
1 - x &= x + x^2 \\
0 &= x^2 + 2x - 1.
\end{aligned}$$

Hence, the fixed points are $-1 \pm \sqrt{2}$. Every other point in $\mathbb{R} \smallsetminus \{-1\}$ has minimum period 2.

The fact that every orbit is 2-periodic implies that that neither fixed point is attracting or repelling. *[This paragraph is tricky, so be lenient with the details.]* To see why, let's pick any open ball U around p. I'll argue that U is neither a basin of attraction or a region of repulsion for p. Find a point x which is not fixed, but is close enough to p that both x and $H(x)$ are in U. Since x is 2-periodic, its orbit never leaves U, so U can't be a region of repulsion for p. On the other hand, since x is periodic but not fixed, its orbit doesn't have a limit, so U can't be a basin of attraction for p.

4 Sweeping the other way

Consider the following dynamical system.

State space: $\mathbf{2}^{\mathbb{N}}$.

Dynamical map: Each `0` that comes after a `1` turns into a `1`.

Let's call this map B. As a demonstration, here's what B does to one point in $\mathbf{2}^{\mathbb{N}}$.

$$\begin{aligned}
w &= \texttt{00110001101110001001011}\ldots \\
B(w) &= \texttt{0011\underline{1}0011\underline{1}111\underline{1}001\underline{1}01\underline{1}11}\ldots
\end{aligned}$$

The changed digits are underlined.

a. Describe all the fixed points of B.

b. Classify each fixed point as attracting, repelling, or neither.

Solution

Sorry about the notation conflict between B for the dynamical map and $B_x(r)$ for the ball around x of radius r!

a. The map B changes every `0` that comes after a `1`, and doesn't change anything else. Hence, the fixed points of B are the sequences in which `10` never occurs. Using the usual bar notation, these sequences are

$$p_n = \underbrace{\mathtt{000\ldots0}}_{n}\overline{\mathtt{1}} \qquad \text{for } n \geq 0$$
$$q = \overline{\mathtt{0}}$$

b. For each $n \geq 0$, the fixed point p_n is attracting, with the open ball $B_{p_n}(2^{-n})$ as a basin of attraction. To see why, consider any point $w \in B_{p_n}(2^{-n})$. The sequences x and p_n match for at least the first $n+1$ digits:

$$w = \underbrace{\mathtt{000\ldots0}}_{n}\mathtt{1}\,.$$

We can use this to predict what $B^k(x)$ will look like out to k digits past the initial `1`.

$$B^k(w) = \underbrace{\mathtt{000\ldots0}}_{n}\underbrace{\mathtt{111\ldots1}}_{k+1}\,.$$

Now we see that $B^k(w)$ matches p_n for at least the first $n+k+1$ digits, so $d(p_n, B^k(w)) \leq 2^{-(n+k+1)}$. Hence, the orbit of w stays in $B_{p_n}(2^{-n})$ forever, and $\lim_{k\to\infty} B^k(w) = p_n$. Since w could've been any point in $B_{p_n}(2^{-n})$, this shows that $B_{p_n}(2^{-n})$ is a basin of attraction for p_n.

The fixed point q is neither attracting or repelling. *[This paragraph is tricky, so be lenient with the details.]* To see why, consider any point $w \in \mathbf{2}^{\mathbb{N}}$ other than q. To be different from q, the sequence w must have a `1` in it somewhere, so it looks like

$$\underbrace{\mathtt{000\ldots0}}_{n}\mathtt{1}$$

for some whole number $n \geq 0$. The map B can only change the digits of w that come after the initial `1`, so $B^k(w)$ will always have the form above, for any $k \geq 0$. That means $d(q, B^k(w))$ will always be 2^{-n}. As a consequence, the orbit of w can't limit to q, and it also can't leave any open ball around q that it starts in. This makes it impossible to find a basin of attraction or a region of repulsion for q.

5 Stripes

Consider the following dynamical system.

> **State space:** $\mathbf{2}^{\mathbb{N}}$.
>
> **Dynamical map:** Each `0` that's followed by a `0` turns into a `1`, and each `1` that's followed by a `1` turns into a `0`.

Let's call this map E. As a demonstration, here's what E does to one point in $\mathbf{2}^{\mathbb{N}}$.

$$
\begin{aligned}
w &= \texttt{001110011011110000101110}\ldots \\
E(w) &= \texttt{\underline{1}0\underline{00}1\underline{0}0\underline{0}10\underline{000}1\underline{111}010\underline{00}10}\ldots
\end{aligned}
$$

a. Find two fixed points of E, and convince the grader they're the only two. *(Corrected. The previous version claimed, incorrectly, that there was only one fixed point.)*

b. Find two points with minimum period 2, and convince the grader they're the only ones.

   HINT: You already know what E does to the first digit of each 2-digit block. Figure out what E^2 does to the first digit of each 3-digit block.

c. The two points with minimum period 2 form a 2-periodic orbit. Convince the grader that this orbit is repelling.

   HINT: To show that a 2-periodic orbit is repelling, you pick a point p on the orbit and show that it's a repelling fixed point of E^2. As a first step, consider a point w which first differs from p at the $(n+1)$st digit, and see what E^2 does to the first n digits of w.

Solution

a. The nth digit of $E(w)$ depends on the nth and $(n+1)$st digits of w as shown below.

In w	`00`	`01`	`10`	`11`
In $E(w)$	`1`	`0`	`1`	`0`

   If `00` or `11` occurs in w, then E changes the first digit of that block, so $E(w) \neq w$. On the other hand, if `00` and `11` never occur in w, then E doesn't change any digits, so $E(w) = w$. Hence, the fixed points of E are $\overline{\texttt{01}}$ and $\overline{\texttt{10}}$—the only two sequences in which `00` and `11` never occur.

b. The 2-periodic points of E are the fixed points of E^2. The nth digit of $E^2(w)$ depends on the nth through $(n+2)$nd digits of w as shown below.

In w	`000`	`001`	`010`	`011`	`100`	`101`	`110`	`111`
In $E(w)$	`11`	`10`	`01`	`00`	`11`	`10`	`01`	`00`
In $E^2(w)$	`0`	`1`	`0`	`1`	`0`	`1`	`0`	`1`

If one of the "bad blocks" 001, 011, 100, and 110 occurs in w, then E changes the first digit of that block, so $E^2(w) \neq w$. On the other hand, if the "good blocks" 000, 010, 101, and 111 are the only three-digit blocks that appear in w, then $E^2(w) = w$. If you try to lay good blocks side by side without creating any bad blocks, the only sequences you can make are $\overline{0}$, $\overline{1}$, $\overline{01}$, and $\overline{10}$. These are the fixed points of E^2, and thus the 2-periodic points of E. Excluding the fixed points we found in part a, we see that $\overline{0}$ and $\overline{1}$ are the only points with minimum period 2.

c. A repelling 2-periodic point of E is defined to be a repelling fixed point of E^2. Our goal, then, is to show that $\overline{0}$ is a repelling fixed point of E^2. I'll argue that $B_{\overline{0}}(2^{-1})$ is a region of repulsion for $\overline{0}$. Consider any point $w \in B_{\overline{0}}(2^{-1})$ other than $\overline{0}$. The sequences w and $\overline{0}$ must match for at least the first two digits, but w must have a 1 in it somewhere, so w must start with a string of 0s followed by a 001. Consulting the table from part b, we see that

$$\begin{aligned} w &= \texttt{000}\ldots\texttt{0}\,\texttt{001} \\ E^2(w) &= \underbrace{\texttt{000}\ldots\texttt{0}}_{n}\,\texttt{1} \end{aligned}$$

with $n \geq 0$. Repeating this argument, we see that the orbit of w eventually contains a point that looks like

$$\texttt{01}$$

or

$$\texttt{10},$$

which is outisde the ball $B_{\overline{0}}(2^{-1})$. Since w could've been any point $w \in B_{\overline{0}}(2^{-1})$ other than $\overline{0}$, this shows that $B_{\overline{0}}(2^{-1})$ is a region of repulsion for $\overline{0}$.