and see that

$$[L_w] = \sum_{y \leq w} (-1)^{l(y)-l(w)} P_{y,w}(1)[M_y]$$

and to study K_0, can do this just as well in category of perverse sheaves. Reduced to a question about composition series in $Perv_B(G/B)$.

Statement and Pf of Kazhdan-Lusztig Conjecture

I. Hecke algebras

W - Coxeter gp, S - set of simple reflections.

Let $\tilde{H}$ be the $\mathbb{Z}[q]$ - algebra ~~defined~~ w/ basis $\{T_w\}_{w \in W}$, relations

$$T_w T_{w'} = T_{ww'} \quad \text{if } l(ww') = l(w) + l(w')$$

$$(T_s + 1)(T_s - q) = 0 \quad \forall s \in S.$$

leave on until Hecke done

So $T_w T_s = (q-1) T_w + q T_{ws}$ if ~~...~~ $ws < w$

If q is specialized to prime power and W is a Weyl gp then $\tilde{H} \otimes_{\mathbb{Z}[q]} \mathbb{C}_q$ ~~...~~ is the algebra of ~~...~~ endomorphisms ~~operators~~ of functions on $G(\mathbb{F}_q)/B(\mathbb{F}_q)$.

So decomposing the rep of $G(\mathbb{F}_q)$ is the same as decomposing reg. rep.

of $\tilde{H} \otimes_{\mathbb{Z}[q]} \mathbb{C}$.

Say: Note the connection w/ being a function on G/B and trivial

central character ($\lambda = 0$).

If $A = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$, then

$$H := \tilde{H} \otimes_{\mathbb{Z}[q]} A$$

is isom. to $\mathbb{Z}[W]$ via $q \mapsto 1$.

Def. $a \in A$, define $\bar{a}$ via $\overline{q^{\frac{1}{2}}} = q^{-\frac{1}{2}}$, and $\overline{(\ )}: H \to H$ by

$$\overline{\sum_w a_w T_w} = \sum \bar{a}_w T_{w^{-1}}^{-1}.$$

Thm. [Kazhdan-Lusztig 1979]

~~There is a uni~~ $\exists!$ basis $\{C'_w\}_{w \in W}$ of H s.t.

1). $\overline{C'_w} = C'_w$

2). $C'_w = q^{-\frac{l(w)}{2}} \sum_{x \in W} P_{x,w}(q^{\frac{1}{2}}, q^{-\frac{1}{2}}) T_x$

w/ $P_{x,w}$ actually in $\mathbb{Z}[q] \subset A$. s.t.

- $P_{x,w} = 0$ unless $x \leq w$
- $P_{w,w} = 1$
- $x \leq w$, $P_{x,w} \in \mathbb{Z}[q]$, of degree $\leq \frac{1}{2}(l(w) - l(x) - 1)$

e.g. $C'_1 = T_1$

$C'_s = q^{-\frac{1}{2}}(T_s+1)$.

Def. $P_{x,w}$ are the Kazhdan-Lusztig polynomials.

Rk. [KL79] works w/ two bases $\{C_w\}$ and $\{C'_w\}$ related by an involution. $\{C'_w\}$ ~~more of~~ ~~comy~~ more natural geometrically, and may now write C_w for C'_w. Soergel and others write $\underline{H}_w$ for C'_w.

Sketch of pf.

existence C'_1, C'_s as above.

If $l(w) > 1$, put $w = vs$, $l(v) = l(w) - 1$. By induction have C'_v.

$$C'_v \cdot C'_s = q^{-\frac{l(v)}{2}} q^{-\frac{1}{2}} \left(T_v + \sum_{y<v} P_{y,v} T_y \right) (T_s + 1)$$

$$= q^{-\frac{l(w)}{2}} \left(T_w + T_v + \sum_{y<v} P_{y,v} \underbrace{T_y T_s}_{} + \sum_{y<v} P_{y,v} T_y \right)$$

~~all length ≤ l(v)~~
~~all of the~~
expand to involve any T_z, $z < vs$.

$$= q^{-\frac{l(w)}{2}} \left(T_w + \sum_{z<w} Q_{z,w} T_z \right) \quad Q_{z,w} \in \mathbb{Z}[q].$$

Can see by induction that $\deg Q_{z,w} \le \frac{1}{2}(\ell(w)-\ell(z))$. Let

$\mathbb{Z} \ni a_{z,w} =$ $q^{\frac{\ell(w)-\ell(z)}{2}}$ – degree coefficient.

Then

$$C'_w := C'_v \cdot C'_s - \sum_{z<w} a_{z,w} C'_z$$

satisfies the criteria.

<u>Uniqueness</u>. Spse we have basis D_w. Then

$$C'_w - D_w = q^{-\frac{\ell(w)}{2}} \sum_{y<w} d_{y,w} T_y$$

$(\in \mathbb{Z}[q],\ \deg \le \frac{1}{2}(\ell(w)-\ell y)\} - 1)$

Let z be maximal w/ $d_{z,w} \neq 0$.

$$C'_w - D_w = q^{\frac{\ell(z)-\ell(w)}{2}} d_{z,w} C'_z + \sum_{\substack{y<w \\ y \not\leq z}} (d_{y,w} - d_{z,w} P_{y,z}) T_y$$

LHS is fixed and $\overline{(\ )}$, so so is RHS.

Fact: $X = \mathrm{Span}\{T_y \mid y<w,\ y \not\leq z\}$ is $\overline{(\ )}$-stable.

So $\left(q^{\frac{\ell(z)-\ell(w)}{2}} d_{z,w} - q^{\frac{\ell(w)-\ell(z)}{2}} \overline{d_{z,w}}\right) C'_z \in X$.

~~$\in$ Span $\{$~~

LHS coeff. $\neq 0$, but $C'_z \notin X$. $\rightarrow\leftarrow$

□

$\hookrightarrow$ degree considerations, would equality get $-1=1$. $\rightarrow\leftarrow$

Conjecture (K-L. 79). All the coeffs. of $P_{y,w}$ are nonnegative.

~~the integers $P_{y,w}(1)$ are nonnegative~~. Moreover,

$$ch\, L_w = \sum_{y \le w} (-1)^{\ell(w)-\ell(y)} P_{y,w}(1) \cdot ch\, M_y$$

$$ch\, M_w = \sum_{y \le w} P_{w_0 w, w_0 y}(1)\, ch\, L_y$$

Rk. Known from the beginning that $P_{y,w}$ ~~tract to~~ related to "failure of local Poincaré duality on Schubert varieties"

In our language, i.e. shifted const. sheaf was Verdier self dual

i.e. ~~Schubert~~ " " " was IC

i.e. ~~" " "~~

Schub. varieties smooth.

II. Categorification

Induction spaces $H \circlearrowright X$, $H \subset G$ closed.

Defn

$G \times^H X =$ geom. quotient of $G \times X$

by H-action $h \cdot (g,x) = (gh^{-1}, hx)$.

Thm. (Quotient equiv.) $G \curvearrowright X$, $H \triangleleft G$ s.t. X is principal H-variety. Then

$$\pi^{\dagger} \circ \mathrm{Infl}_{G/H}^{G} : D^b_{G/H}(\cancel{X/H}) \xrightarrow{\sim} D^b_G(X).$$

~~Thm (induction equivalence) Notation as above, let~~

~~$i : X \longrightarrow G \times^H X$. $\forall\, \mathcal{F} \in D^b_G(G \times^H X)$, have nat. isom.~~

~~$x \longmapsto [(e,x)]$~~

~~$i^* \mathrm{For}^G_H(\mathcal{F})[-\dim G/H] \simeq i^! \mathrm{For}^G_H [\dim G/H]$~~

~~inducing t-exact equiv. of cats~~

~~$D^b_G(G \times^H X) \simeq D^b_H(X)$.~~

Flag variety

Recall $T \subset B \subset G$, $W = N_G(T)/T$, $\{s_\alpha\}_{\alpha \in \Delta} = S$ simple reflections

Can write $w = s_{i_1} \cdots s_{i_k}$, $l(w) = k$ reduced expression.

$$G = \bigsqcup_w B\dot{w}B \implies G/B = \bigsqcup_w B\dot{w}B/B =: \mathcal{B}$$

write $\mathcal{B}_w = B\dot{w}B/B$ Schubert cells, $\overline{\mathcal{B}_w} = \bigsqcup_{y \leq w} B\dot{y}B/B$

Schub. variety. $\mathcal{B}_w \simeq \mathbb{C}^{l(w)}$.

Gives good stratification; then Bruhat stratification.

$\leadsto (j_S)_* \mathcal{L} \in D^b_S(X, \mathbb{C})$.

$D^c_B(\mathcal{B})$ B-equivariant derived cat

$D^c_{(B)}(\mathcal{B}) \subset D^b_c(X)$ full subcat. constructible wrt

Bruhat stratification.

<u>Notation</u>. $IC_w := IC(\overline{\mathcal{B}_w})$.

For $s_\alpha \in S$ simple reflection,

$$\overline{\mathcal{B}_{s_\alpha}} = \mathbb{C} \sqcup \{pt\} = \mathbb{P}^1$$

$$\text{So } IC_{s_\alpha} \simeq i_{s*} \underline{\mathbb{C}}_{\overline{\mathcal{B}_s}}[1], \quad i_s : \overline{\mathcal{B}_s} \hookrightarrow \mathcal{B}$$

<u>Convolution</u>

We will make $D^b_B(\mathcal{B})$ into a monoidal cat.

<u>Motivation</u>. Say f, g are $B \times B$-equivt fns. on G. Then

$$(f * g)(\cancel{x}xb) = \int_G f(y) g(xby^{-1}) dy$$

do on pallel boards

$$= \int_G f(yb) g(xbb^{-1}y^{-1}) dy$$

$$= \int_G f(y) g(xy^{-1}) dy \qquad (*)$$

$$= (f * g)(x).$$

To study functions on G/B, needed ~~other~~ B-equivt on the other side. ~~#~~

Say. This is why we needed $D^b_B(\mathcal{B})$.

Def. $\ast : D^b_B(\mathcal{B}) \times D^b_B(\mathcal{B}) \longrightarrow D^b_B(\mathcal{B})$

$$(\mathcal{F}, \mathcal{G}) \longmapsto \mathcal{F} \ast \mathcal{G}$$

defined by

$$(g, hB) \longmapsto ghB$$

$$\begin{tikzcd} G/B \times G/B & G \times G/B \arrow[l, "p"', "B\times B\text{-equivt, smooth}"] \arrow[r, "q"] & G \times^B G/B \arrow[r, "\text{mult}"] & G/B \end{tikzcd}$$

$G \times G/B$ with $B \times B$ acting:

$$(b_1, b_2)(g_1, g_2B) = (b_1 g_1 b_2^{-1}, b_2 g_2 B)$$

by quotient equivalence

$$(\mathcal{F}, \mathcal{G}) \longmapsto \mathcal{F} \boxtimes \mathcal{G} \longmapsto p^{\dagger}(\mathcal{F} \boxtimes \mathcal{G}) \longmapsto \left(q^{\dagger}\, \mathrm{Infl}_B^{B\times B}\right)^{-1}(p^{\dagger} \mathcal{F} \boxtimes \mathcal{G})$$

(*)

$$\left(q^{\dagger}\, \mathrm{Infl}_B^{B\times B}\right)^{-1}(p^{\dagger} \mathcal{F} \boxtimes \mathcal{G}) = \mathcal{F} \tilde{\boxtimes} \mathcal{G} \longmapsto m_*(\mathcal{F} \tilde{\boxtimes} \mathcal{G}) = \mathcal{F} \ast \mathcal{G}.$$

$\boxed{\text{Note (*)-steps are what use equivariance } \underline{\text{in}} \text{ some cases}}$

On level of functions:

$$(f \ast g)(x) = \int_{\substack{m^{-1}(x) \\ = \{(g_1, g_2) \mid g_1 g_2 = x\}}} f(p_1(h_1, h_2))\, g(p_2(h_1, h_2))$$

$$= \int_{\{h_2 = h_1^{-1}x\}} f(h_1)\, g(h_1^{-1}x),$$

Same as for functions!

Skyscraper
"

The identity for convolution is the δ-function $\delta_1 = IC_1$.

<u>lem.</u> $\forall\, \mathcal{F} \in D^b_B(X)$, $\exists$ nat. isoms $IC_1 * \mathcal{F} \simeq \mathcal{F} \simeq \mathcal{F} * IC_1$.

<u>Pf.</u> $\because$ IC_1 is supported on $B/B = pt$, we can compute w/

$$\mathcal{B}_1 \times G/B \xleftarrow{p} B \times G/B \xrightarrow{q} B \times^{B} G/B \xrightarrow{m = id} G/B$$

typo in Achar

and $\mathcal{B}_1 \times G/B \simeq B \times^{B} G/B$, $p = q$.

(can use B $\because$ consider the supports of the pullbacks).

$$IC_1 * \mathcal{F} = m_* (q^\dagger)^{-1} p^\dagger \mathcal{F} \simeq \mathcal{F}.$$

$\square$

<u>lem.</u> $\exists$ nat. isom $(\mathcal{F} * \mathcal{G}) * \mathcal{H} \simeq \mathcal{F} * (\mathcal{G} * \mathcal{H})$

<u>Pf.</u> Draw a 4×4 commutative grid, use base change thms many times.

$\square$

~~In the proof, actually show~~

In the proof, actually show can use

$$\underbrace{G/B \times \cdots \times G/B}_{k} \leftarrow G \times \cdots \times G \times G/B \to G \times^{B} \cdots \times^{B} G \times^{B} G/B \to G/B$$

to compute $\mathcal{F}_1 * \cdots * \mathcal{F}_k$.

<u>Thm.</u> $D^b_B(\mathcal{B})$ is monoidal cat $*$.

<u>Pf.</u> Pentagon and triangle axioms use $k = 3, 4$ are ~~[illegible]~~ of

$f^\dagger \mathbb{D} = f^*[d]\,\mathbb{D} = f^*\mathbb{D}[-d] = \mathbb{D}f^![-d] = \mathbb{D}f^\dagger$

Recall $\mathrm{Semis}_B(\mathcal{B}, \mathbb{Q}) \subset D^b_B(\mathcal{B}, \mathbb{Q})$ is the additive cat of finite $\oplus$ of shifted perse sheaves.

Prop. $\mathrm{Semis}_B(\mathcal{B}, \mathbb{Q})$ closed under $*$.

Pf. Each functor used preserves $\mathrm{Semis}_B(\mathcal{B}, \mathbb{Q})$: $\boxtimes$ commutes w/ IC, $p^\dagger, q^\dagger$ smooth pullbacks, m proper so use decomp. thm. $\square$

Lem. $\mathbb{D}\mathcal{F} * \mathbb{D}\mathcal{G} \simeq \mathbb{D}(\mathcal{F} * \mathcal{G})$.

Pf. $\mathbb{D}$ commutes w/ $\boxtimes$, ~~smooth~~ $\dagger$-pullback, m (proper). $\square$

Lem. $\forall s \in S$, have a Cartesian square

$$\begin{tikzcd} G \times^B P_s/B \arrow[r, "m"] \arrow[d, "pr_1"] & G/B \arrow[d, "\pi_s"] \\ G/B \arrow[r, "\pi_s"] & G/P_s \end{tikzcd}$$

P_s = corresponding parabolic

So by proper base change, $\mathcal{F} * IC_s \simeq \pi_s^* \pi_{s*} \mathcal{F}[1]$.
~~In particular, triangulated operation~~

Pf. ~~Formal, besides~~ Formal.

Rk. On functions,

$$\begin{tikzcd} G \arrow[d, "\pi_s"] \\ G/P_s \end{tikzcd}$$

$$(\pi_s)^*\pi_{s*} f(x) = \pi_* f(\pi(x)) = \int_{\{y \mid \pi_s(y) = \pi_s(x)\}} f(y) = \int_{\{y \mid y \in xP_s\}} f(y) = \int_{y \in G} f(y)\, \chi_{P_s}(x^{-1}y),$$

and $\overline{\mathcal{B}}_s$ is smooth. $\square$

This generalizes to the Bott-Samelson varieties

Let $s_1, \dots, s_k$ be simple reflections,

$$\mathcal{B}(s_1,\dots,s_k) = P_{s_1} \times^B P_{s_2} \times^B \cdots \times^B P_{s_k}/B \xrightarrow{m} \mathcal{B}$$

$$(p_1, \dots, p_k B) \longmapsto p_1 \cdots p_k B.$$

Lem. $IC_{s_1} * \cdots * IC_{s_k} \simeq m_* \mathbb{C}_{\mathcal{B}(s_1,\dots,s_k)}[k]$ is a s.simple complex.

Pf. $IC_{s_i} \simeq \mathbb{C}_{\overline{\mathcal{B}}_{s_i}}[1]$ is constant so $IC_{s_1} \tilde{\boxtimes} \cdots \tilde{\boxtimes} IC_{s_k} \simeq \mathbb{C}_{\mathcal{B}(s_1,\dots,s_k)}[k]$

Now use decomp. thm. $\square$

Sorry Rk. There are ways to do all this in pos. char, see Achar. This is one place things start to break down.

Prop. Image$(m) = \overline{\mathcal{B}_w} \subset \mathcal{B}$, and $m^{-1}(\mathcal{B}_w) \xrightarrow{m \sim} \mathcal{B}_w$,

so $IC_{s_1} * \cdots * IC_{s_k}$ supported on $\overline{\mathcal{B}_w}$ and restricts to

$\mathbb{C}_{\mathcal{B}_w}[\ell(w)]$. on $\mathcal{B}_w$.

Pf. First part omit but note

$m^{-1}(\mathcal{B}_w) \simeq \mathcal{B}_w$ and $IC * \cdots * IC \simeq m_* \mathbb{C}_{\mathcal{B}(s_1,\dots,s_w)}[k]$

$\Rightarrow$ last claim. $\square$

recall define $\tau^{\leq 0}$, $\tau^{\geq 0}$ "resolution-wise" idea functorial all $\mathcal{F}(U \to X)$ and $\mathcal{F}_X$.

<u>Cor.</u> $w = s_1 \cdots s_k$ red. exp. Then

$IC_w(\mathbb{Q})$ occurs w/ mult. 1 in $\mathcal{F} = IC_{s_1}(\mathbb{Q}) * \cdots * IC_{s_k}(\mathbb{Q})$.

<u>Pf.</u> Know $\mathcal{F}$ is s. simple complex supported on $\overline{B_w}$ s.t. $\mathcal{F}|_{B_w} = \underline{\mathbb{Q}}_{B_w}[l(w)]$.

this must be the restriction of an indecomposable summand of $\mathcal{F}$.

the only simple this can be is IC_w. □

We are finding the Hecke algebra!! (Bruhat decomp is how we remember the order on W).

<u>Parity</u> $\mathcal{F} \in D^b_{\mathcal{B}}(\mathcal{B})$ or $D^b_{(\mathcal{B})}(\mathcal{B})$ <u>$*$-even</u> if $H^i(j_w^* \mathcal{F}) \simeq \underline{\mathbb{C}}_{B_w}[l(w)]$ $\oplus n_w$

for i even, zero for i odd. <u>$*$-odd</u> if $\mathcal{F}[1]$ is $*$-even.

Likewise <u>$!$-even/odd</u>. ($\mathcal{F}$ is $*$/$!$-odd/even iff $D\mathcal{F}$ is $!$/$*$-odd/even)

<u>Rk.</u> Notion depends only on $\mathrm{For}(\mathcal{F}) \in D^b_c$, $\because$ 6-functors intertwine For.

<u>Lem.</u> Cat. of $*$-even objects in $D^b_{\mathcal{B}}(\mathcal{B})$ or $D^b_{(\mathcal{B})}(\mathcal{B})$ gen under extensions by
$(j_w)_! \underline{\mathbb{Q}}_{B_w}[l(w)]$

$!$ even ~~$*$-odd~~ by $j_{w*} \underline{\mathbb{Q}}_{B_w}[l(w)]$.

<u>Pf.</u> Induct on dim supp and 2-out-of-3 property for distinguished triangles.

lem. $\mathcal{F}, \mathcal{G} \in D^b_B(\mathcal{B}, \mathbb{Q})$ or $D^b_{(B)}(\mathcal{B}, \mathbb{Q})$. If $\mathcal{F}$ is $*$-even and $\mathcal{G}$ is $!$-odd, $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = 0$ and vice versa.

Pf. ETS for $\mathcal{F} = j_{w!}\underline{\mathbb{Q}}_{\mathcal{B}_w}[2n]$, $\mathcal{G} = j_{w*}\underline{\mathbb{Q}}_{\mathcal{B}_w}[2m+1]$.

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}(j_{w!}\underline{\mathbb{Q}}_{\mathcal{B}_w}[2n], j_{w*}\underline{\mathbb{Q}}_{\mathcal{B}_w}[2m+1])$$

$$\simeq \mathrm{Hom}(\underbrace{j_v^* j_{w!}}_{\substack{=0 \text{ if} \\ v \neq w.}}\underline{\mathbb{Q}}_{\mathcal{B}_w}, \underline{\mathbb{Q}}_{\mathcal{B}_v}[2m+1-2n])$$

if $v = w$, $j_v^* j_! = id$, and get

$$H^{2m-2n+1}(\mathbb{C}^{\ell(w)}; \underline{\mathbb{C}}) = 0.$$

odd!

For equivt category same but w/ equivt coho. $\square$

lem. $$H^i((j_{w!}\underline{\mathbb{Q}}_{\mathcal{B}_w} * IC_s)|_{\mathcal{B}_v}) = \begin{cases} \mathbb{Q} & \text{if } ws > w,\ i = -1,\ v \in \{w, ws\} \\ \underline{\mathbb{Q}} & \text{if } ws < w,\ i = 1,\ v \text{ " "} \\ 0 & \text{otherwise} \end{cases}$$

Pf. Assume $ws > w$.

$w = s_1 \ldots s_k$, $s_k \neq s$.

$\rightarrow$

In Bruhat decomp we actually have

$$B_w = U_{s_1}\dot{s}_1 \times \dots \times U_{s_k}\dot{s}_k \simeq \mathbb{A}^k$$

$$B_{ws} = U_{s_1}\dot{s}_1 \times \dots \times U_{s_k}\dot{s}_k \times U_s\dot{s} \simeq \mathbb{A}^{k+1}$$

$$Y := B_w P_s/P_s = \pi_s(B_w) = \pi_s(B_{ws}) \subset G/P_s.$$

$\underset{\mathbb{P}^1}{P_s} = \underset{pt}{B} \cup \underset{\mathbb{A}^1}{U_s \dot{s} B}$, so $\pi_s|_{B_w} : B_w \xrightarrow{\sim} B_w P_s/P_s$

$$P_s = B \cup U_s\dot{s}B = B \cup B\dot{s}B$$

Consider

$$\begin{tikzcd} G/B \arrow[d, "\pi_s"] \\ G/P_s \end{tikzcd}$$

$$\pi_s^{-1}(b\dot{w}P_s/P_s) = b\dot{w}B/B \cup b\dot{w}B\dot{s}B/B \simeq \mathbb{P}^1$$

So just need to see what B-orbits the fibres lie in. If $ws > w$, then $BwBsB = BwsB$, and

claim if $ws > w$, $\pi_s|_{B_w}$ is isom.

$ws < w$ $\pi_s|_{B_w}$ is $\mathbb{A}^1$-fibration

See $\pi_s^{-1}(b\dot{w}P_s/P_s) = b\dot{w}B/B \cup b\dot{w}\dot{s}B/B$

So $|_{B_w}$, injective $\Rightarrow$ isom.

If $ws < w$, put $w' = ws$: see that $\mathbb{A}^1$-part of fibre lies in $BwB = Bw'sB$.

$$\begin{tikzcd} B_w \arrow[r, hook, "j_w"] \arrow[d, "\pi_s"] & \mathcal{B} \arrow[d, "\pi_s"] \\ Y \arrow[r, hook, "i"] & G/P_s \end{tikzcd}$$

not proper

$$\pi_{s*} j_{w!} \mathbb{Q}_{B_w} = i_!(\pi_s|_{B_w})_! \mathbb{Q}_{B_w} = i_! \mathbb{Q}_Y$$

~~Let~~ $\cancel{\pi_s}$

$$\begin{tikzcd}
\pi_s^{-1}(Y) = B_w \cup B_{ws} \arrow[r, hook, "h"] \arrow[d, "\pi|_s|_{\pi^{-1}(Y)}"'] \arrow[dr, phantom, "\ulcorner"] & B \arrow[d, "\pi_s"] \\
Y \arrow[r, hook, "i"] & G/P_s
\end{tikzcd}$$

So $\underbrace{\pi_s^{*}\pi_{s*}}_{(-\,*\,IC_s)} j_{w!}\underline{\mathbb{Q}}_{B_w} = \pi_s^{*}\, i_!\, \underline{\mathbb{Q}}_Y \simeq h_!\, \underline{\mathbb{Q}}_{\pi_s^{-1}(Y)}$ (base change)

$\because$ $h_!$ is extending zero,

$$H^i\left(\left(\cancel{IC_s *}\, j_{w!}\underline{\mathbb{Q}}_{B_w} * IC_s\right)\Big|_{B_v}\right) = \begin{cases} \mathbb{Q} & i=0,\ v\in\{w, ws\} \\ 0 & \text{otherwise} \end{cases}.$$

Similar since π_s is $\mathbb{A}^1$-fibration.

$\square$

<u>Prop.</u> $\mathcal{F} \in D^b_B(B)$ $*$-even, $\mathcal{F} * IC_s$ is $*$-odd

and vice-versa.

<u>Pf.</u> Showed for $j_{w!}\underline{\mathbb{Q}}_{B_w}[2n]$, and they generate. $\square$

<u>Cor.</u> $w \in W$. If $l(w)$ is even, $IC_w(\mathbb{Q})$ is $*$-even and $!$-even. If $l(w)$ is odd, then $IC_w(\mathbb{Q})$ is $*$-odd, $!$-odd.

$K_{\oplus}(\mathrm{Semis}_B(\mathcal{B}, \mathbb{Q}))$ is a ring.

<u>Thm.</u>

$$ch_q : K_{\oplus}(\mathrm{Semis}_B(\mathcal{B}, \mathbb{Q})) \xrightarrow{\sim} \mathcal{H}$$

$$ch_q([\mathcal{F}]) = \sum_{\substack{w \in W \\ i \in \mathbb{Z}}} (\mathrm{rk}\, H^i(\mathcal{F}|_{\mathcal{B}_w}))\, q^{\frac{i}{2}} T_w$$

s.t.

I) $ch_q[\mathcal{F}[1]] = q^{-\frac{1}{2}} ch([\mathcal{F}])$

II) $ch_q(D(\mathcal{F})) = \overline{ch_q([\mathcal{F}])}$

III) $ch_q(IC_w(\mathbb{Q})) = C'_w$.

Note ch_q makes sense on $D^b_B(\mathcal{B})$. will need this.

<u>Pf.</u> <u>Step 1.</u> ~~Eq~~ I) obvious, can use to define $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-module structure, ch is still hom.

$$ch_q(\underbrace{[IC_w]}_{\substack{\text{basis,} \\ \text{supp. on } \overline{\mathcal{B}_w}}}) \in \underbrace{q^{-\frac{\ell(w)}{2}} T_w + \sum_{y < w} \mathbb{Z}[q^{\pm\frac{1}{2}}] T_y}_{\text{upper triangular} \Rightarrow \text{isom.} \checkmark}$$

<u>Step 2.</u> Additive on triangles of $*$-even or odd objects:

$*$-even $\Rightarrow$ les in coho breaks into ses's, ranks then add $\checkmark$
for $\mathcal{F}|_{\mathcal{B}_w} \to \mathcal{F}'|_{\mathcal{B}_w} \to \mathcal{F}''|_{\mathcal{B}_w}$

<u>Step 3.</u> ch_q is multiplicative on $D^b_B(\mathcal{B})$, ~~*-even~~: $\mathcal{F}$ - $*$-even, then

$ch_q([\mathcal{F}] * IC_s) = ch_q[\mathcal{F}]\, ch_q IC_s$.

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to$$

all $*$-even. Have seen

$$\mathcal{F}' * IC_s \to \mathcal{F} * IC_s \to \mathcal{F}'' * IC_s \to$$

triangle and all $*$-odd. ~~By step~~

Step 2 $\Rightarrow$ $ch_q(\mathcal{F} * IC_s) = ch_q(\mathcal{F}' * IC_s) + ch_q(\mathcal{F}'' * IC_s)$

$$ch_q \mathcal{F} = ch_q \mathcal{F}' + ch_q \mathcal{F}''$$

~~so if holds~~ so if claim holds for $\mathcal{F}', \mathcal{F}''$, holds for $\mathcal{F}$:

$$\left(\quad ch_q(\mathcal{F} * IC_s) = (ch_q \mathcal{F}' + ch_q \mathcal{F}'') ch_q IC_s. \right.$$

ETS for $\mathcal{F} = j_{w!} \underline{\mathbb{Q}}_{\mathcal{B}_w} [2n]$, $w \in W$, $n \in \mathbb{Z}$.

we know

$$ch_q(j_{w!} \underline{\mathbb{Q}}_{\mathcal{B}_w} [2n] * IC_s) = \begin{cases} q^{-n-\frac{1}{2}} (T_w + T_{ws}) & ws > w \\ q^{-n+\frac{1}{2}} (T_w + T_{ws}) & ws < w \end{cases}$$

OTOH clearly $ch_q(j_{w!} \underline{\mathbb{Q}}_{\mathcal{B}_w} [2n]) = q^{-n} T_w$

$$ch_q(IC_s) = q^{-\frac{1}{2}} (T_s + 1). \quad \checkmark$$

<u>Step 4</u>. $ch_q(\mathcal{F} * \mathcal{G}) = ch_q \mathcal{F} \cdot ch_q \mathcal{G}$ $\forall\, \mathcal{F}, \mathcal{G} \in Semis_B(\mathcal{B}, \mathbb{Q})$

formal.

$\mathbb{Z}$-alg. hom,

Step 5. $\because$ $\mathbb{D}$ commutes w/ $*$, induces $d: K_\oplus \to K_\oplus$ ~~ring hom.~~

$\mathbb{D}(\mathcal{F}[1]) = (\mathbb{D}\mathcal{F})[-1] \Rightarrow d(q^{\frac{1}{2}}[\mathcal{F}]) = q^{-\frac{1}{2}} d([\mathcal{F}])$

$\mathbb{D} IC_s = IC_s \Rightarrow d([IC_s]) = d([IC_s])$.

We know $ch_q IC_s = q^{-\frac{1}{2}}(T_s+1)$, so d corresp. to $\mathcal{H} \to \mathcal{H}$
$q^{\frac{1}{2}} \mapsto q^{-\frac{1}{2}}$
$C'_s \mapsto C'_s$

must be $\overline{(\ )}$. ✓

Step 6

We have $ch([IC_w]) = C'_w$.

5) $\Rightarrow$ $\overline{ch_q([IC_w])} = ch_q([IC_w])$. $\forall x, w \in W$,

~~$P'_{x,w} \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$~~

$$P'_{x,w} := \sum_{i \in \mathbb{Z}} \left(\mathrm{rk}\, H^i(j_x^* IC_w)\right) q^{\frac{i+l(w)}{2}} \in \mathbb{Z}[q^{\pm\frac{1}{2}}],$$

so that

$$ch_q([IC_w]) = q^{-\frac{l(w)}{2}} \sum_{x \in W} P'_{x,w} T_x$$

know $P'_{x,w} = 0$ if $x \not\leq w$

$P'_{w,w} = 1$ $\quad x < w$, $H^i(j_x^* IC_w(a)) = 0$

if $-l(w) \leq i \leq -l(x) - 1$; ~~first bound general for purity~~ 2nd

and j^* is t. exact in std t-structure

Perv on $\overline{B_w}$

property of IC's (usual coho. sits in strict) $\Rightarrow \in \mathbb{Z}[q^{\pm 1}]$.

know $H^i = 0$ unless $i \equiv l(w)$ (mod 2)

together $\Rightarrow P'_{x,w} \in \mathbb{Z}[q]$, degree $\leq \frac{1}{2}(l(w) - l(x) - 1)$. $\to$ D

So $ch_q(IC_w)$ must be C'_w.

$\square$

Cor. from Step 6,

$$P_{x,w} = \sum_{i \in \mathbb{Z}} \operatorname{rk} H^i(IC_w(\mathbb{Q})[-\ell(w)]|_{B_x})\, q^{\frac{i}{2}}$$

$$\in \mathbb{Z}_{\geq 0}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}].$$

Other parts of conjecture

Thm. $ch : K_0(D^b_B(\mathcal{B}, \mathbb{Q})) \xrightarrow{\sim} \mathbb{Z}[W]$

$$[\mathcal{F}] \longmapsto \sum_{\substack{w \in W \\ i \in \mathbb{Z}}} (-1)^i \operatorname{rk} H^i(\mathcal{F}|_{\mathcal{B}_w})\, w.$$

$$ch([j_{w!}\underline{\mathbb{Q}}_{\mathcal{B}_w}]) = w$$

$$ch([IC(\mathbb{Q})) = C'_w|_{q^{\frac{1}{2}} = -1}.$$

Therefore

$$ch[IC(\mathbb{Q})] = \sum_{x \in W} (-1)^{\ell(w)-\ell(x)} P_{x,w}|_{q=1}\, [j_{w!}\underline{\mathbb{Q}}_{\mathcal{B}_w}[\ell(w)]].$$

Pf.

$$\begin{tikzcd}
K_{\oplus}(\mathrm{Semis}_{B}(\mathcal{B}, \mathbb{Q})) \arrow[r, "ch_q", "\sim"'] \arrow[d] & \mathcal{H} \arrow[d, "q^{\frac{1}{2}} \mapsto -1"] \\
K_0(D^b_B(\mathcal{B}, \mathbb{Q})) \arrow[r, "ch"] & \mathbb{Z}[W]
\end{tikzcd}$$

$\Rightarrow$ ch is surjective

$\Rightarrow$ isom $\because$ both free abelian. rk ~~???~~ $\# W$.

Statements for ch follow from ch_q.

ch is ring hom. follows from ~~ch~~ ch_{q}; ETS for IC's.

last claim, specialize def of C'_w and apply ch^{-1}.

~~[illegible]~~

$\cdot\ IC_w * IC_v \longmapsto$ ~~C_w~~ $\cdot C'_v$

$\downarrow$

$C_w C'_v|_1 \cdot C'_v|_1 \cdot C'_v|_1$

~~[illegible]~~

Recap on how we got here:

$$\mathcal{O}_0 \hookrightarrow D^{rh}_{B}(G/B)^B \xrightarrow{R.H.} Perv_B(G/B, \mathbb{C})$$

M_w	~~$j_{w*!}\mathcal{O}_{B_w}$~~ $j_{w!}\mathcal{O}_{B_w}$	$j_{w!}\mathbb{C}_{B_w}$
L_w	$j_{w*!}\mathcal{O}_{B_w}$	IC_w
$M_w^\vee$	$j_*\mathcal{O}_{B_w}$	$j_{w*}\mathbb{C}_{B_w}$

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$\mathfrak{gl}_n$

$\{$ nilpotent matrices $\} \hookrightarrow \{$partitions of $n\} \hookleftarrow$ representations of S_n

Now G – connected reductive, w/ $T \subset B \subset G$ fixed.

$\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$