Part II Equivariant derived category (1)

Q. Why equivariant sheaves?

~~Kazhdan~~

A. Hecke algebras

• ~~Geometric~~ Classical Satake isom

$\cancel{C_c^{\infty}(G(\mathbb{F}_q[[t]])\backslash G(}$

$$H = C_c^{b}\left(G(\mathbb{F}_q((t)))/G(\mathbb{F}_q[[t]])\right)^{G(\mathbb{F}_q[[t]])} \longleftrightarrow \cancel{\mathbb{F}(\mathbb{F}_q((t)))/\mathbb{F}(\mathbb{F}_q[[t]])}$$

functions ~~on~~ $\left(T(\mathbb{F}_q((t)))/T(\mathbb{F}_q[[t]]) = X_*(T)\right.$ $X^*(T^\vee)$ $\left.\to \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]\right)$

w/ image in W-invariants, i.e.

$$H \xrightarrow{\sim} R(G^\vee) \otimes \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}].$$

Geometric Satake: replace ~~sheaves~~ fns. w/ perverse sheaves, rep ring w/ rep cat

$$\mathrm{Perv}_{G(\mathcal{O})}\left(G(\mathbb{C}((t)))/G(\mathbb{C}[[t]]), *\right) \xrightarrow{\sim} (\mathrm{Rep}(G^\vee), \otimes)$$

equiv. of monoidal cats.

Kazhdan-Lusztig theory

family of statements along the lines that there are ring homs

⊕ (shifts of pervs) $\cdot\, \chi: K_{\oplus}(D^b_{m,B}(G(\mathbb{C})/B(\mathbb{C}), \bar{\mathbb{Q}}_\ell)) \twoheadrightarrow$ ~~fns~~ functions $B(\mathbb{F}_q)\backslash G(\mathbb{F}_q)/B(\mathbb{F}_q) \to \bar{\mathbb{Q}}_\ell$

$\cdot\, ch: K_{\oplus}(\mathrm{Semis}_B \subset D^b_B(G/B)) \xrightarrow{\sim}$ Hecke algebra

Sometimes convolution takes us out of the heart.

Reminder on equivariant sheaves

Def. $G \circlearrowright X$ topo gp. $\circlearrowright$ topo. space. A G-equivt sheaf is $(\mathcal{F}, \theta)$ where $\mathcal{F} \in Sh(X)$ and if

$$\begin{array}{ccc} G \times X & \xrightarrow{a} & X \\ \downarrow p & & \\ X & & \end{array}$$

$$\theta : p^*\mathcal{F} \xrightarrow{\sim} a^*\mathcal{F}.$$

s.t. if

$$\begin{array}{ccccc} (g,h,x)\ G \times G \times X & \xrightarrow{m} & G \times X & \xrightarrow{a} & X \\ \swarrow p_{23} \qquad \searrow b & & \downarrow p & & \\ (h,x)\ G \times X \qquad G \times X\ (g,hx) & & X & & \\ \swarrow a \quad \searrow p & & & & \\ X \qquad X & & & & \end{array}$$

$(g,h,x) \mapsto (g,hx)$

$m^*p^*\mathcal{F} \longrightarrow m^*a^*\mathcal{F}$

~~$m^*\theta : m^*a^*\mathcal{F} \longrightarrow m^*p^*\mathcal{F}$~~

is equal to

~~$b^*\theta \circ p_{23}^*\theta : p_{23}^*a^*\mathcal{F} \longrightarrow p_{23}^*p^*\mathcal{F} \longrightarrow$~~

$$G \times G \times X \overset{m}{\underset{p_{23}}{\overset{b}{\rightrightarrows}}} G \times X \underset{a}{\overset{p}{\rightrightarrows}} X$$

$$m^*\theta = b^*\theta \circ p_{23}^*\theta \quad \text{(composable} \because \text{we have an action)}$$

on stalks this means

~~$\theta_{(gh,x)} : \mathcal{F}_x \longrightarrow \mathcal{F}_{ghx}$~~

~~$(m^*\theta\mathcal{F})_{(g,h,x)} = \theta\mathcal{F}_{(gh,x)} \; (gh,x) \longrightarrow \mathcal{F}_x$~~

is equal to

$$(m^*\theta)_{(g,h,x)} : (m^*p^*\mathcal{F})_{(g,h,x)} \longrightarrow (m^*a^*\mathcal{F})_{(g,h,x)}$$

$\| \qquad\qquad\qquad \|$

$\mathcal{F}_x \qquad\qquad\qquad \mathcal{F}_{ghx}$

~~$\longrightarrow \mathcal{F}_{hx} \xrightarrow{\theta_{(g,hx)}} \mathcal{F}_{ghx}$~~

is equal to

$$(p_{23}^*\theta)_{(g,h,x)} : \underset{\mathcal{F}_x}{\underset{\parallel}{(p_{23}^* p^*\mathcal{F})_{(g,h,x)}}} \longrightarrow \cancel{(p_{23}^*\theta)_{(g,h,x)}}\ \underset{\mathcal{F}_{hx}}{\underset{\parallel}{(p_{23}^* a^*\mathcal{F})_{(g,h,x)}}}$$

followed by

$$(b^*\theta)_{(g,h,x)} : \underset{\mathcal{F}_{hx}}{\underset{\parallel}{(b^* p^*\mathcal{F})_{(g,h,x)}}} \longrightarrow \underset{\mathcal{F}_{ghx}}{\underset{\parallel}{(b^* a^*\mathcal{F})_{(g,h,x)}}}.$$

A <u>morphism</u> of equivariant sheaves is a morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ s.t.

$$\begin{tikzcd} p^*\mathcal{F} \arrow[r, "\theta_{\mathcal{F}}"] \arrow[d, "p^*\phi"'] & a^*\mathcal{F} \arrow[d, "a^*\phi"] \\ p^*\mathcal{G} \arrow[r, "\theta_{\mathcal{G}}"] & a^*\mathcal{G} \end{tikzcd}$$

$Sh_G(X)$ is an abelian category.

$a, p : G\times X \to X$ are both smooth of rel dim. $\dim G$

$$\left(a = G\times X \xrightarrow{\sim} G\times X \xrightarrow{p} X,\quad (g,x) \overset{\sim}{\longmapsto} (g, gx) \longmapsto gx \right)$$

so define $a^\dagger = a^*[\dim G]$, $p^\dagger[\dim G] : Perv(X) \to Perv(X\times G)$.

So G-equivariant perverse sheaf is $(\mathcal{F}, \theta)$ w/

$$\theta : p^\dagger\mathcal{F} \xrightarrow{\sim} a^\dagger\mathcal{F}$$

and same cocycle condition.

$Perv_G(X)$ is abelian.

<u>Rk.</u> When θ exists for perv $\mathcal{F}$, it's unique (structure, not prop).

Not true of such G-structure.

~~With study more Ph~~

- $\mathrm{Perv}_G(X)$ and $\mathrm{Sh}_G(X)$ will be hearts of a ~~tri~~ triangulated cat $D^b_G(X) (\neq D^b(\mathrm{Sh}_G$

- Some functors (more later)

$H \subset G$ cl. subgp, $X \circlearrowleft G$.

$$H \times X \xrightarrow{i \times id_X} G \times X \underset{p}{\overset{a}{\rightrightarrows}} X$$

$(\mathcal{F}, \theta) \in \mathrm{Perv}_G(X)$, then

$$(i \times id_X)^* \theta[-\dim(G/H)]$$

gives $\mathcal{F}$ an H-equivt structure. Thus we get

$$\mathrm{For}^G_H : \mathrm{Perv}_G(X) \longrightarrow \mathrm{Perv}_H(X)$$

Forgetful functor

If H trivial, ~~get~~ get $\mathrm{Perv}_G(X) \longrightarrow \mathrm{Perv}(X)$.

$H \trianglelefteq G$ normal, then

$$G \times X \xrightarrow{\pi \times id_X} G/H \times X \underset{p}{\overset{a}{\rightrightarrows}} X$$

can define

$$\mathrm{Infl}^G_{G/H} : \mathrm{Perv}_{G/H} \longrightarrow \mathrm{Perv}_G$$

$$(\mathcal{F}, \theta) \longmapsto (\mathcal{F}, \pi^* \theta[\dim G/H]).$$

If H connected, this is equiv. of cats.

Pullback

$f : X \longrightarrow Y$ smooth G-equivt

$\theta : p^\dagger \mathcal{F} \to a^\dagger \mathcal{F}$

$$\begin{tikzcd} G \times X \arrow[r, "id \times f"] \arrow[d, "p{,}\, a"'] \arrow[dr, phantom, "\ulcorner"] & G \times Y \arrow[d, "p{,}\, a"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

$\mathcal{F} \in \mathrm{Perv}_G(Y)$ claim $f^\dagger \mathcal{F} \in \mathrm{Perv}_G(X)$ naturally. using $(id \times f)^\dagger \theta$

$(id \times f)^\dagger \theta = 2f^\dagger ~~p_*~~ p^\dagger \mathcal{F} \to 2f^\dagger ~~p_*~~ a^\dagger \mathcal{F}$

Seems to just use commutativity of two diagrams

$(f \circ pr_2)^\dagger = (pr_2 \circ id \times f)^\dagger$

$$\begin{array}{c} pr_2^\dagger f^\dagger = (id \times f)^\dagger \\ \Big\downarrow (id \times f)^\dagger \\ (id \times f)^\dagger a^\dagger \\ \| \\ a^\dagger f^\dagger \end{array}$$

Prop. In this case

$f^\dagger : \mathrm{Perv}_G(Y) \longrightarrow \mathrm{Perv}_G(X)$

faithful. As usual fully faithful when fibres connected.

Def. A G-variety X is principal if $G \curvearrowright X$ freely and geom. quotient exists.

Thm. X - principal, then $\mathrm{Perv}_G(X) \simeq \mathrm{Perv}(X/G)$.

Rk. Proof proceeds via checking descent data, approaches a non-point stuff we almost reached last year

$f : X \longrightarrow Y$

desc. datum is for $\mathcal{F}$

$\phi : p_2^* \mathcal{F} \xrightarrow{\sim} p_1^* \mathcal{F}$ in $Sh(X \times_Y X)$

+ cocycle condition.

Useful fact

Lie G = Lie(G)

($d_1 \mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$)

$\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$

$g \mapsto (d\, \mathrm{Ad}_g)$

$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$

X-homogeneous for G. Then all G-equivt Perv sheaves are shifted local systems, and

$$\mathrm{Loc}^{ft}_G(X) \simeq \mathbb{C}[\underbrace{G^x/(G^x)^0}_{\text{"equivt fundamental gp"}}]\text{-mod}^{fg}$$

G^x = $\mathrm{Stab}_G(x)$, $(G^x)^0$ = identity compt.

Pf. A perv. sheaf whose cohom sheaves are ~~constant~~ local systems must be a shifted local system, and equivariance + homogeneity spreads $H^i(\mathcal{F})$ being loc. constant on some open U (by constructibility) to being a local system

ETS for IC sheaves, probably, can induct on length, have filtration by IC's.

Equivariant derived category

- Cannot take pairs $(\mathcal{F}, \theta)$, $\mathcal{F} \in D^b_c(X)$. Turns out not triangulated!
- Cannot take $D^b Sh_G$ or $D^b Perv_G$, no faithful functor!

Want:

- Triangulated cat $D^b_G(X)$ w/ heart $= Perv_G(X)$
- t-exact $For: D^b_G(X) \longrightarrow D^b(X)$ restricting to $Perv_G \longrightarrow Perv$
- 6-functors intertwining For.

~~Example~~

Acyclic resolutions

Def. $f: X \to Y$ is (universally) n-acyclic if $\forall\, Y' \to Y$, the map $f': X \times_Y Y' \longrightarrow Y'$ and $\forall\, \mathcal{F}$ on Y',

$$\mathcal{F} \longrightarrow {}^p\tau^{\le n} f'_* f'^* \mathcal{F} \qquad (f_* = \underline{\quad}$$

is Isom. ⊛

acyclic if 0-acyclic, ∞-acyclic if n-acyclic $\forall n$.

Say: About fibres not picking up too much cohomology

e.g. Z s.t. $H^i(Z; \mathbb{C}) = 0$ $\forall$ $0 \le i \le n$, then $X \times Z \longrightarrow X$ is n-acyclic.

~~Fact~~ Def. G-Map $\mathcal{U} \longrightarrow X$ from principal G-variety $\mathcal{U}$ is a G-resolution of X. acyclic resolution if acyclic.

Facts
- Composite of acyclic is acyclic
- acyclic resolutions exist $\forall n$.:

~~Use~~ let $p: P \longrightarrow X$ be any resolution of X.

We will define a ~~triangulated~~ category $D^{\mathrm{I}}_G(X,P)$ and then put $D^b_G(X) := \lim_{\mathrm{I}} D^{\mathrm{I}}_G(X,P)$ as the resolutions $P \to X$ are taken to be n-acyclic for $n \to \infty$.

$D^b_G(X)$ will be triangulated but $D^{\mathrm{I}}_G(X,P)$ won't!

$\exists$ also simplicial definition, OG-m... but historical reasons...

Def. Given

$$X \xleftarrow{p} U \xrightarrow{q} U/G$$

define $\mathrm{Obj}(D^b_G(X,P)) = \{ (\mathcal{F}_X, \underbrace{\mathcal{F}(U \to X)}_{=\mathcal{F}} \in D_G(U/G)$

$$\beta: p^* \mathcal{F}_X \xrightarrow{\sim} q^* \mathcal{F}(U \to X) \}$$

$$\mathrm{Mor}_{\cancel{D^b_G(M,P)}}(\mathcal{F}, \mathcal{G}) = \{ \alpha = (\alpha_X, \alpha_U) \mid \alpha_X : \mathcal{F}_X \to \mathcal{G}_X$$

$$\alpha_U : \mathcal{F}(U \to X) \to \mathcal{G}(U \to X)$$

$$\beta \circ p^*(\alpha_X) = q^*(\alpha_U) \circ \beta$$

e.g. X - free, $D^b_G(X,X) = D^b_G(X/G)$

$G = \{e\}$, $D^b_G(X, U) = D^b(X)$.

$$\mathrm{For}: D^b_G(X, U) \longrightarrow D^b(X).$$

$$\mathcal{F} \longmapsto \mathcal{F}_X$$

Def let $U \xrightarrow{\nu} V$ be morphism of G-spaces.

$$\begin{tikzcd} U \arrow[rr, "\nu"] \arrow[dr, "p"'] & & V \arrow[dl, "r"] \\ & X & \end{tikzcd}$$

$$\bar{\nu}: U/G \longrightarrow V/G$$

$$\nu^*: D^b_G(X, V) \longrightarrow D^b_G(X, U)$$

$$(\mathcal{F}_X, \mathcal{F}(V \to X), \beta) \longmapsto (\mathcal{F}_X, \bar{\nu}^* \mathcal{F}(V \to X), \gamma)$$

$\mathcal{F}(V \to X)$

$$\gamma = \nu^*(\beta): p^* \mathcal{F}_X = \nu^* r^* (\mathcal{F}_X) \to \nu^* q^* (\mathcal{F}(V \to X)) = q^* \bar{\nu}^* \ldots$$

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Prop. [See Bernstein-Lunts] Let $I = [a,b] \subset \mathbb{Z}$

Define $D^I_G(X,\mathcal{U}) \subset D^b_G(X,\mathcal{U})$ to be the full subcat of objects $\mathcal{F}$ s.t. $\mathcal{F}_X \in D^I_G(X)$. i.e. $\mathcal{F}_X$ has perverse cohomology only in degrees in the interval I.

Then if $\mathcal{U}$ is n-acyclic,

$$q^*: D^I(\mathcal{U}/G) \xrightarrow{\sim} D^I_G(X,\mathcal{U})$$

is equiv. of cats.

Corollary.

If $\mathcal{U} \to X$ is n-acyclic, $n > |I|$, then

$$\begin{tikzcd}
S/G \arrow[r, "\bar p"] & \mathcal{U}/G \\
S = \mathcal{U}\times_X V \arrow[u, "p_S"] \arrow[r, "pr_\mathcal{U}"] \arrow[d] & \mathcal{U} \arrow[u, "q_\mathcal{U}"'] \arrow[d, "u"] \\
V \arrow[r] & X
\end{tikzcd}$$

now given $\mathcal{V}$ n-acyclic, $n > |I|$ and V such relation, set

$C_{V,\mathcal{U}} : (pr_\mathcal{U}^*)^{-1} \circ pr_V^*$

$: D^I_G(X,\mathcal{U}) \xrightarrow{\sim} D^I_G(X,S) \xrightarrow{\sim} D^I_G(X,V)$.

will take limit of this system

$$D^I(X,\mathcal{U}) \simeq D^I(X,S)$$

$$(\mathcal{F}_X, \mathcal{F}_\mathcal{U}, \beta) \longmapsto (\mathcal{F}_X, (q_\mathcal{U}^*\mathcal{F}_\mathcal{U}, \bar p^*\mathcal{F}_\mathcal{U},$$

$\beta : q_\mathcal{U}^*\mathcal{F}_\mathcal{U} \to u^*\mathcal{F}_X$

$\gamma : pr_\mathcal{U}^* q_\mathcal{U}^* \mathcal{F}_\mathcal{U} \to p_S^* \bar p^* \mathcal{F}$

so this data gives

$(\mathcal{F}_X, \bar p^* \mathcal{F}_\mathcal{U} \in D^I(S/G))$, isom

$p_S^* \bar p \mathcal{F}_\mathcal{U} \xrightarrow{\sim} s\, q_\mathcal{U}^* \mathcal{F}_\mathcal{U} \xrightarrow{\sim} s^* u^* \mathcal{F}_X$

~~some~~ facts, along w/ $X \times Z \to X$ example.

X - smooth, connected s.t. $H^k(X;\mathbb{Z}) = 0 \ \forall\, 0 < k \leq n$, H^{n+1} free abelian. Then $X \to pt$ is n-acyclic.

the $C_{V,\mathcal{U}}$ obey cocycle condition etc. then need to take limits. Let's unpack:

Reason: $G \hookrightarrow GL_n$, and for GL_n Stiefel mflds provide the resolutions, ~~so~~ thanks to bounds on their coho.

Bernstein-Lunts defined this cat in ~~___~~ as a ~~limit~~ colimit of simpler categories. This is the limit (?lg-lg?) definition written naked.

<u>Def.</u> A G-<u>equivariant complex</u> is the data

- $\mathcal{F}_X \in D^b_c(X)$
- $\forall\ \mathcal{U} \xrightarrow{j} X$ acyclic resolution, $\mathcal{F}(\mathcal{U} \to X) \in D^b_c(\mathcal{U}/G)$ ($\pi_{\mathcal{U}}: \mathcal{U} \to \mathcal{U}/G$)

 w/ $\beta_{\mathcal{U}}: j^+\mathcal{F}_X \xrightarrow{\sim} \pi_{\mathcal{U}}^+ \mathcal{F}(\mathcal{U} \to X)$

- $\forall$ pairs of resolutions,

$$\begin{tikzcd}
\mathcal{U}_1 \times_X \mathcal{U}_2 \arrow[rr, "p_2"] \arrow[d, "p_1"] \arrow[dr] & & \mathcal{U}_2 \arrow[dr] & \\
\mathcal{U}_1 \arrow[dr] & (\mathcal{U}_1 \times_X \mathcal{U}_2)/G \arrow[rr, "\overline{p_2}"] \arrow[d, "\overline{p_1}"] & & \mathcal{U}_2/G \\
& \mathcal{U}_1/G & &
\end{tikzcd}$$

an isom $\phi: (\overline{p_1})^+\mathcal{F}(\mathcal{U}_1 \to X) \xrightarrow{\sim} (\overline{p_2})^+\mathcal{F}(\mathcal{U}_2 \to X)$.

or limit system condition

"agree on overlaps" "$\times_X^{\#} = \cap$"

this is not a topology on G-varieties, but it ~~__~~ behaves a lot like one.

- Fix a resolution $P: \mathcal{U} \to X$. Define $D_G(X,P)$ as triples $(\mathcal{F}_X, \mathcal{F}(\mathcal{U}\to X), \beta)$, $\in D^b(\mathcal{U}/G)$

$\beta: j^*\mathcal{F}_X \xrightarrow{\sim} q^*\mathcal{F}(\mathcal{U}\to X)$

q = quotient map. Mor: $(\alpha_X, \overline{\alpha})$ $\alpha_X: \mathcal{F}_X \to \mathcal{G}_X$, $\overline{\alpha}: \mathcal{F}(\mathcal{U}\to X) \to \mathcal{G}(\mathcal{U} \to X)$

$\beta \circ p^*(\alpha_X) = q^*(\overline{\alpha}) \circ \beta$

acyclic

take lim, j: i- mo

leave on

• These Φ obey a (hard to write down) cocycle condition ~~about triple intersections.~~ Morally about $\ell_1 \to \ell_2 \to \ell_3$

$$\begin{tikzcd} \ell_1 \arrow[r] \arrow[rr, bend right] & \ell_2 \arrow[r] & \ell_3 \end{tikzcd}$$

A <u>morphism</u> $\Psi : \mathcal{F} \to \mathcal{G}$ is the data

• $\Psi_X : \mathcal{F}_X \to \mathcal{G}_X$ in $D^b_c(X)$

• $\forall\, j : \mathcal{U} \to X$, $\Psi_{\mathcal{U}} : \mathcal{F}(\mathcal{U} \to X) \to \mathcal{G}(\mathcal{U} \to X)$ in $D^b_c(\mathcal{U}/G)$.

~~Obeying a condition on "overlaps" $\forall\, j : \mathcal{U} \to X$.~~

~~$\forall\, j_1 : \mathcal{U}_1 \to X,\ j_2 : \mathcal{U}_2 \to X,$~~

$\forall\, j$ s.t.

$$\begin{tikzcd} j^+ \mathcal{F}_X \arrow[r, "j^+ \Psi_X"] \arrow[d, "\beta_{\mathcal{U}}"', "\simeq"] & j^+ \mathcal{G}_X \arrow[d, "\sim"', "\beta_{\mathcal{U}}"] \\ \pi_{\mathcal{U}}^+ \mathcal{F}(\mathcal{U} \to X) \arrow[r, "\pi_{\mathcal{U}}^+ \Psi_{\mathcal{U}}"] & \pi_{\mathcal{U}}^+ \mathcal{G}(\mathcal{U} \to X) \end{tikzcd}$$

and on "overlaps" $\forall\, j_1 : \mathcal{U}_1 \to X$, $j_2 : \mathcal{U}_2 \to X$,

$$\begin{tikzcd} \overline{p}_1^{\,+} (\mathcal{F}(\mathcal{U}_1 \to X)) \arrow[r, "p_1^+ \Psi_{\mathcal{U}_1}"] \arrow[d, "\phi_{\mathcal{F}}"'] & p_1^+ \mathcal{G}(\mathcal{U}_1 \to X) \arrow[d, "\phi_{\mathcal{G}}"] \\ \overline{p}_2^{\,+} \mathcal{F}(\mathcal{U}_2 \to X) \arrow[r, "p_2^+ \Psi_{\mathcal{U}_2}"] & p_2^+ \mathcal{G}(\mathcal{U}_2 \to X) \end{tikzcd}$$

~~Ex~~. Define $[1] : D^b_G(X) \longrightarrow D^b_G(X)$

by $(\mathcal{F}[1])_X = \mathcal{F}_X[1]$, $(\mathcal{F}[1])(U \longrightarrow X) = \mathcal{F}(U\to X)[1]$

- $\mathcal{F} \to \mathcal{G} \to \mathcal{H} \xrightarrow{(1)} \mathcal{F}[1]$

<u>distinguished</u> in $D^b_G(X)$ if

$$\mathcal{F}(U \to X) \longrightarrow \mathcal{G}(U\to X) \longrightarrow \mathcal{H}(U\to X) \to \mathcal{F}$$

is in $D^b(U/G)$ $\forall$ $j : U \to X$.

(include limit description!)

- For: $D^b_G(X) \longrightarrow D^b(X)$
  $\mathcal{F} \longmapsto \mathcal{F}_X$.

- ${}^pD^b_G(X)^{\geq a} = \{ \mathcal{F} \in D^b_G(X) \mid \mathcal{F}_X \in {}^pD^b_c(X)^{\geq a} \}$

  " " " $\leq a$ = { " " | " " $\leq a$ }

~~and the heart~~

Recap.

<u>Thm.</u> $D^b_G(X)$ is a triangulated category w/ a triangulated functor For: $D^b_G(X) \longrightarrow D^b(X)$. The heart of above t-structure is $\mathrm{Perv}_G(X)$.

<u>Rk.</u> Can do std t-structure also.

This seems horrible! But we can work w/ a smaller class of resolutions, and define everything using just them. ~~[illegible]~~ This set will be cofinal wrt acyclicity.

Def. Collection $(j_\alpha : U_\alpha \to X)_{\alpha \in I}$ is <u>acyclic covering</u> if ~~$\forall n$,~~ contains an n-acyclic covering $\forall n \geq 0$.

let $\pi_\alpha : U_\alpha \to U_\alpha/G$. ~~An acyclic glui.~~

<u>Acyclic glueing datum</u> wrt fixed covering is

$$\left\{ \begin{array}{l} \mathcal{F}_X \in D^b_c(X),\ \mathcal{F}(U_\alpha \to X) \in D^b_c(U_\alpha/G) \\ \beta_\alpha : j_\alpha^+ \mathcal{F}_X \xrightarrow{\sim} \pi_\alpha^* \mathcal{F}(U_\alpha \to X) \quad \forall \alpha \\ \phi_{\alpha,\beta} : \bar{p}_1^+ \mathcal{F}(U_\alpha \to X) \longrightarrow \bar{p}_2^+ \mathcal{F}(U_\beta \to X) \quad \forall \alpha, \beta \end{array} \right\}$$

Satisfying same conditions as before.

Thm. Given acyclic covering, The functor

$$D^b_G(X) \longrightarrow \mathrm{Glue}((U_\alpha \to X))$$

$$\mathcal{F} \longmapsto \text{glueing datum for } \mathcal{F} \text{ wrt chosen cov}$$

is equiv. of cats

(~~proof~~ key lemma: if $j : U \to X$ is n acyclic, then $\forall k$

if asked

$$D^b_G(X)^{[k,k+n]} \longrightarrow D^b(U/X)^{[k,k+n]}$$

condition on $\mathcal{F}_X$

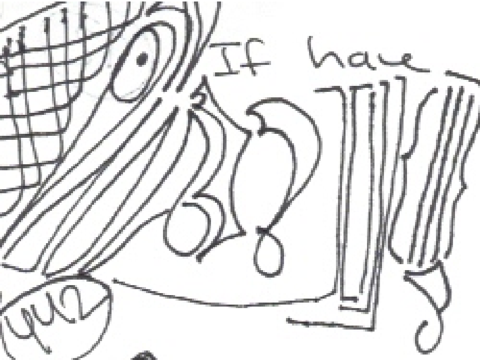
$$\mathcal{F} \longmapsto \mathcal{F}(U \to X)$$

is fully faithful, ~~as~~ triangulated (and converse holds as at triangles)

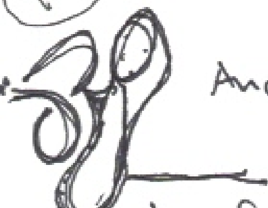
<u>Summary</u>. To describe $\mathcal{F}$, need

- $\mathcal{F}(U \to X)$ for all $U \to X$ in a big enough set $\in D^b_c(U/G)$
- $\forall$ morphism of resolutions, ~~a map~~ an isom $\alpha_\varphi : \varphi^* \mathcal{F}(V \to X) \xrightarrow{\sim} \mathcal{F}(U \to X)$

$$\begin{tikzcd} U \arrow[rr, "\varphi"] \arrow[dr] & & V \arrow[dl] \\ & X & \end{tikzcd}$$



If have
$$\begin{tikzcd} U \arrow[r, "\varphi"] \arrow[dr] & V \arrow[r, "\psi"] \arrow[d] & W \arrow[dl] \\ & X & \end{tikzcd}, \qquad \alpha_\psi \circ \varphi^*\alpha_\varphi = \alpha_{\psi\varphi}.$$



And that's it! No need to check all other coherence isoms!

Sheaf functors. Recall that "all functors commute w/ smooth pullback!" Let $\mathcal{F}, \mathcal{G} \in D^b_G(X)$

Fix an acyclic cover $\{j_\alpha : U_\alpha \to X\}$

$\otimes$: $(\mathcal{F} \otimes \mathcal{G})_X = \mathcal{F}_X \otimes \mathcal{G}_X$, $(\mathcal{F}\otimes\mathcal{G})(U_\alpha \to X) = \mathcal{F}(U_\alpha \to X) \otimes \mathcal{G}(U_\alpha \to X)$

$R\mathcal{H}om$: $R\mathcal{H}om(\mathcal{F},\mathcal{G})_X = R\mathcal{H}om(\mathcal{F}_X, \mathcal{G}_X)$

$R\mathcal{H}om(\mathcal{F},\mathcal{G})(U_\alpha \to X) = R\mathcal{H}om(\mathcal{F}(U_\alpha \to X), \mathcal{G}(U_\alpha \to X))$.

~~Given~~ ~~do base-change argument to show is well-defined~~...

$\beta_\alpha : j_\alpha^\dagger \mathcal{F}_X \xrightarrow{\sim} \pi_\alpha^\dagger \mathcal{F}(U_\alpha \to X)$

$\gamma_\alpha : j_\alpha^\dagger \mathcal{G}_X \xrightarrow{\sim} \pi_\alpha^\dagger \mathcal{G}(U_\alpha \to X)$

induces

$$\begin{array}{ccc} j_\alpha^\dagger \mathcal{F}_X \otimes j_\alpha^\dagger \mathcal{G}_X & \xrightarrow{\sim} & \pi_\alpha^\dagger \mathcal{F}(U_\alpha \to X) \otimes \pi_\alpha^\dagger \mathcal{G}(U_\alpha \to X) \\ \wr\wr & & \parallel \\ j_\alpha^\dagger(\mathcal{F}_X \otimes \mathcal{G}_X) & & \pi_\alpha^\dagger(\mathcal{F}(U_\alpha \to X) \otimes \mathcal{G}(U_\alpha \to X)) \end{array}$$

etc.

$f : X \to Y$ G-morphism

~~$f_*\mathcal{F}$~~ $\mathcal{F} \in D^b_G(X)$, $f_*\mathcal{F}$ def'd by $For(f_*\mathcal{F}) = f_*\mathcal{F}_X$.

$U \to Y$ acyclic,
$$\begin{tikzcd} & X \arrow[r, "f"] & Y \\ U \times_Y X \arrow[ur] \arrow[rr, "f_U"] \arrow[d] & & U \arrow[u] \arrow[d] \\ (U\times_Y X)/G \arrow[rr, "\bar{f}_U"] & & U/G \end{tikzcd}$$
$f_*\mathcal{F}(U \to Y) = \bar{f}_{U*}(\mathcal{F}(U \times_Y X \to X))$

same for $f_!$.

f^* and $f^!$.

$$\mathrm{For}(f^*\mathcal{F}) := f^*\,\mathrm{For}(\mathcal{F})$$

for acyclic resolutions of X of the form

$$\begin{tikzcd} \mathcal{U} \times_Y X \arrow[r] \arrow[d] & \mathcal{U} \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

set $f^*(\mathcal{U} \times_Y X \to X) = \bar{f}_{\mathcal{U}}^{\,*}\mathcal{F}(\mathcal{U} \to Y)$

$\bar{f}_{\mathcal{U}} : D^b_c(\mathcal{U}/G) \to D^b_c(\mathcal{U}\times_Y X/G)$

But base-change of an acyclic cover is an acyclic cover, so this suffices.

Thm. All the properties the 6 functors have (adjunctions, triangles etc) lift to $D^b_G(X)$.

Pf. The identities all commute w/ smooth base change.

Inflation and ~~Restriction~~ partial fgthfl functors

- this and const sheaves and $\mathbb{D}$, following [BL94]. $\boxed{\text{Todo}}$

Constant sheaves

Let $M \in \mathcal{O}[G/G^0]$-mod $= \mathrm{Perv}_G(\mathrm{pt.})$ ~~set~~ $X \xrightarrow{a_X} \mathrm{pt}$, set

$\underline{M}_X = a_X^* M$. Equiv't derived sheaf.

~~Set ω~~ $\omega_X = a_X^! \mathbb{C}$, $\mathbb{D}(\mathcal{F}) = R\mathcal{H}om(\mathcal{F}, \omega_X)$. Behaves as usual.