References: Mirkovic-Vilonen (2007) (MV)
Baumann-Riche (2017) (BR)
Beilinson-Drinfeld preprint 'Quantization of Hitchin's integrable system and Hecke eigensheaves'
Pramod's Book

Notation:

G complex connected reductive group

$G^\vee$ Langlands dual group

k field of char 0

Note: Can do this for k commutative Noetherian ring of finite gl. di which is what MV does

$$\mathcal{O} = \mathbb{C}[[t]], \quad \mathcal{K} = \mathbb{C}((t)), \quad Gr = G(\mathcal{K})/G(\mathcal{O})$$

Geometric Satake Equivalence (MV, 2007):

$$P_{G(\mathcal{O})}(Gr, k) \simeq Rep_k(G^\vee) \quad \text{as tensor categories}$$

Tannakian Formalism

Theorem: $\mathcal{C}$ abelian k-linear category equipped with:

- exact k-linear faithful functor
$$F : \mathcal{C} \to Vect_k \quad (\text{fiber functor})$$
- k-bilinear functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (tensor product)
- $\mathcal{U} \in \mathcal{C}$ (unit)
- $\phi_{X,Y,Z} : X \otimes (Y \otimes Z) \xrightarrow{\sim} (X \otimes Y) \otimes Z$ natural in X, Y, Z
- $\lambda_X : \mathcal{U} \otimes X \xrightarrow{\sim} X \xleftarrow{\sim} X \otimes \mathcal{U} : \rho_X$ natural in X
- $\psi_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$ natural in X and Y
- $\tau_{X,Y} : F(X \otimes Y) \xrightarrow{\sim} F(X) \otimes F(Y)$ natural in X and Y

s.t.
- $F(\phi_{X,Y,Z}), F(\lambda_X), F(\rho_X), F(\psi_{X,Y})$ satisfy usual associativity, unit, commutativity conditions
- if $\dim_k(F(X)) = 1$, then $\exists X^* \in \mathcal{C}$ s.t. $X \otimes X^* \simeq \mathcal{U}$ (guarantee H has inverse, else H affine monoid scheme)

Then $\exists$ affine group scheme H s.t. F admits a canonical factorization

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\tilde{F}"] \arrow[dr, "F"'] & Rep_k(H) \arrow[d, "\text{Forget}"] \\ & Vect_k \end{tikzcd}$$

s.t. $\tilde{F}$ equivalence of categories respecting tensor product and unit

Proof: See [BR, Section 2]

Affine Grassmannian Review

$B^+, B^- \subset G$ opposite Borels, $T \subset B$ maximal torus, $N^+ \subset B^+, N^- \subset B^-$ unipotent radicals
$X_*(T)$ = lattice of cocharacters
$X_*^+(T)$ = dominant cocharacters

For $\mu \in X_*(T)$, let $L_\mu \in Gr$ be associated point.
For $\lambda \in X_*^+(T)$, let $Gr^\lambda := G(\mathcal{O}) L_\lambda$,

We have $Gr = \coprod_{\lambda \in X_*^+(T)} Gr^\lambda$, $\overline{Gr^\lambda} = \coprod_{\substack{\mu \in X_*^+(T) \\ \mu \leq \lambda}} Gr^\mu$ (i.e. a stratification)

$\dim Gr^\lambda = \langle 2\rho, \lambda \rangle$, open dense in $\overline{Gr^\lambda}$

Iwasawa Decomposition:
For $\mu \in X_*(T)$, let $S_\mu^\pm := N^\pm(K) L_\mu$ semi-infinite orbits

We have $Gr = \coprod_{\mu \in X_*(T)} S_\mu^+ = \coprod_{\mu \in X_*(T)} S_\mu^-$ $\quad \overline{S_\mu^+} = \coprod_{\substack{\nu \in X_*(T) \\ \nu \leq \mu}} S_\nu^+$, $\overline{S_\mu^-} = \coprod_{\substack{\nu \in X_*(T) \\ \nu \geq \mu}} S_\nu^-$

$S_\mu^+ = \{x \in Gr : \lim_{z \to 0} z \cdot x = L_\mu\}$, $S_\mu^- = \{x \in Gr : \lim_{z \to \infty} z \cdot x = L_\mu\}$

Let S be stratification by Gr^λ's.
Theorem: $P_S(Gr, k)$ is semisimple.
Note: Need char $k = 0$ here. Not true for char $k = p$.
Sketch: As Gr^λ simply connected (affine bundles over partial flag varieties), the only simple local system is $\underline{k}$. Let IC_λ be corresponding IC sheaf.

$\Rightarrow$ simple objects of $P_S(Gr, k)$ are of the form IC_λ

Then as $P_S(Gr, k)$ is a finite-length category so suffices to prove:
$\forall \lambda, \mu \in X_*^+(T)$, $\operatorname{Ext}^1(IC_\lambda, IC_\mu) = \operatorname{Hom}(IC_\lambda, IC_\mu[1]) = 0$.

Case 1: $\lambda = \mu$
Consider

$$\begin{tikzcd} Gr^\lambda \arrow[r, hook, "j"] \arrow[dr, hook, "j_\lambda"'] & \overline{Gr^\lambda} \arrow[r, hook, "i"] & Gr \\ & Gr_\lambda \arrow[u, hook, "i_\lambda"'] & \end{tikzcd}$$

$(i_\lambda i)^* IC_\lambda$ is concentrated in negative perverse degrees (IC properties)
$(i_\lambda i)^! IC_\lambda$ " ——— " positive " ——— "
$\Rightarrow \operatorname{Hom}((i_\lambda i)^* IC_\lambda, (i_\lambda i)^! IC_\lambda[1]) = 0$

Apply $\mathrm{Hom}^\bullet((i_\lambda)_!(-), IC_\lambda[1])$ to triangle

$$j_! j^*(IC_\lambda|_{\overline{Gr^\lambda}}) \to IC_\lambda|_{\overline{Gr^\lambda}} \to i_! i^*(IC_\lambda|_{\overline{Gr^\lambda}}) \to$$

to get

$$\underbrace{\mathrm{Hom}((i_\lambda i)_!(i_\lambda i)^* IC_\lambda, IC_\lambda[1])}_{\substack{\| \\ 0 \text{ by adjunction}}} \to \mathrm{Hom}(IC_\lambda, IC_\lambda[1]) \to \mathrm{Hom}((j_\lambda)_! \underline{k}_{Gr^\lambda}[\dim Gr^\lambda], IC_\lambda[1])$$

$\mathrm{Hom}((j_\lambda)_! \underline{k}_{Gr^\lambda}[\dim Gr^\lambda], IC_\lambda[1])$ $\cong$ (adjunction) $\mathrm{Hom}(\underline{k}_{Gr^\lambda}, \underline{k}_{Gr^\lambda}[1]) \cong H^1(Gr, k) = 0$

as Gr^λ is affine line bundle over partial flag variety so $H^{odd}(Gr) = 0$.

Case 2: $Gr^\lambda \not\subset \overline{Gr^\mu}$ and $Gr^\mu \not\subset \overline{Gr^\lambda}$

Let $i_\mu : \overline{Gr^\mu} \hookrightarrow Gr$. IC_μ supported on $\overline{Gr_\mu}$ $\Rightarrow$ $IC_\mu = (i_\mu)_*(i_\mu)^* IC_\mu$

$$\Rightarrow \mathrm{Hom}(IC_\lambda, IC_\mu[1]) \cong \mathrm{Hom}((i_\mu)^* IC_\lambda, (i_\mu)^* IC_\mu[1])$$

Let $Z = \overline{Gr^\lambda} \cap \overline{Gr^\mu}$. Let $i_Z : Z \hookrightarrow \overline{Gr^\mu}$. As $(i_\mu)^* IC_\lambda$ supported on Z, $(i_\mu)^* IC_\lambda = (i_Z)_! \mathcal{F}$ $\exists \mathcal{F} \in D^b_S(Z, k)$.

$\mathcal{F}$ concentrated in negative perverse degrees, $(i_Z)^!(i_\mu)^* IC_\mu \cong (i_\mu i_Z)^! IC_\mu$ concentrated in positive degrees so the same argument as in Case 1.

Case 3: $\lambda \neq \mu$ and $Gr^\lambda \subset \overline{Gr^\mu}$ or $Gr^\mu \subset \overline{Gr^\lambda}$

WLOG (Verdier duality anti-autoequivalence fixing IC), $Gr^\mu \subset \overline{Gr^\lambda}$.
Consider triangle

$$IC_\mu \to (j_\mu)_*(j_\mu)^* IC_\mu \to \mathcal{F} \to$$

$(j_\mu)_*(j_\mu)^* IC_\mu \cong (j_\mu)_* \underline{k}_{Gr^\mu}[\dim Gr^\mu]$

concentrated in nonnegative perverse degrees

Applying $\mathrm{Hom}(IC_\lambda, -)$, get

$$\mathrm{Hom}(IC_\lambda, \mathcal{F}) \to \mathrm{Hom}(IC_\lambda, IC_\mu[1]) \to \mathrm{Hom}(IC_\lambda, (j_\mu)_* \underline{k}_{Gr^\mu}[\dim Gr^\mu + 1]) = 0$$

$\mathrm{Hom}(IC_\lambda, \mathcal{F}) = 0$ as $\mathcal{F}$ supported on $\overline{Gr^\mu} \subset \overline{Gr^\lambda} \setminus Gr^\lambda$

as $(j_\mu)^* IC_\lambda$ concentrated in deg $< -\dim Gr^\mu$, cohomology in degrees of same parity as $\dim Gr^\lambda$, $\dim Gr^\lambda \equiv \dim Gr^\mu \mod 2$
$\Rightarrow (j_\mu)^* IC_\lambda$ concentrated in deg $\leq -\dim Gr^\mu - 2$

Note that the forgetful functor $P_{G(\mathcal{O})}(Gr,k) \to P_S(Gr,k)$ is fully faithful
Then as $P_S(Gr,k)$ is semisimple, each IC_λ is in the essential image so we get an equivalence of categories.

Dimension Estimates

Proposition: $\mu \in X_*(T)$. Inside $\overline{S_\mu}$, the boundary of S_μ is given by a hyperplane section under an embedding of Gr in projective space
i.e. $\partial S_\mu = \overline{S_\mu} \cap \psi^{-1}(H_\mu)$, ψ embedding, H_μ hyperplane

Theorem: Let $\lambda, \mu \in X_*(T)$ with λ dominant.
Then $\overline{Gr^\lambda} \cap S_\mu^+ \neq \emptyset$ iff $L_\mu \in \overline{Gr^\lambda}$. In this case, $\overline{Gr^\lambda} \cap S_\mu$ has pure dimension $\langle \rho, \lambda+\mu \rangle$. ($\overline{Gr^\lambda} \cap S_\mu^- \neq \emptyset$ iff $L_\mu \in \overline{Gr^\lambda}$. $\overline{Gr^\lambda} \cap S_\mu^-$ pure of dim. $\langle \rho, \lambda-\mu \rangle$).

Proof: As S_μ^+ is attractive variety of L_μ and $\overline{Gr^\lambda}$ is stable under T action, we have $\overline{Gr^\lambda} \cap S_\mu^+ \neq \emptyset$ iff $L_\mu \in \overline{Gr^\lambda}$.

In particular, $\overline{Gr^\lambda} \cap S_\mu^+ \neq \emptyset$ implies $\mu \leq \lambda$. Thus $\overline{Gr^\lambda} \subset \overline{S_\lambda^+}$.
Conjugating by a lift of w_0, we get $\overline{Gr^\lambda} \subset \overline{S_{w_0\lambda}^-}$.
Then $L_\mu \in \overline{Gr^\lambda} \Rightarrow w_0\lambda \leq \mu \leq \lambda$.

Induct on $\langle \rho, \mu - w_0\lambda \rangle$ to show $\dim(\overline{Gr^\lambda} \cap \overline{S_\mu^+}) \leq \langle \rho, \lambda+\mu \rangle$.
If $\mu = w_0\lambda$, then $\overline{Gr^\lambda} \cap \overline{S_{w_0\lambda}^+} \subset \overline{S_{w_0\lambda}^+} \cap \overline{S_{w_0\lambda}^-} = \{L_{w_0\lambda}\}$, dimension 0.
If $\mu > w_0\lambda$, choose a hyperplane H_μ as in Proposition. Let $C \in \operatorname{Irr}(\overline{Gr^\lambda} \cap \overline{S_\mu})$ and and $D \in \operatorname{Irr}(C \cap \psi^{-1}(H_\mu))$.

Then $\dim D \geq \dim C - 1$ but $D \subset \psi^{-1}(H_\mu) \cap \overline{Gr^\lambda} \cap \overline{S_\mu^+} = \partial S_\mu^+ \cap \overline{Gr^\lambda} = \bigcup_{\nu < \mu} S_\nu^+ \cap \overline{Gr^\lambda}$
so by induction, $\dim D \leq \max_{\nu<\mu} \langle \rho, \lambda+\nu \rangle = \langle \rho, \lambda+\mu \rangle - 1$.
$\Rightarrow \dim C \leq \dim D + 1 \leq \langle \rho, \lambda+\mu \rangle$ as desired.

Now for the other inequality. If $\mu = \lambda$, then $\overline{Gr^\lambda} \cap \overline{S_\lambda^+} = \overline{Gr^\lambda}$, irreducible of dim $\langle 2\rho, \lambda \rangle = \langle \rho, \lambda+\lambda \rangle$
so also true for $\overline{Gr^\lambda} \cap S_\lambda^+$.
Assume $\mu < \lambda$. Let $C \in \operatorname{Irr}(\overline{Gr^\lambda} \cap S_\mu^+)$. Set $d := \langle \rho, 2\lambda \rangle - \dim C$ and H_λ hyperplane as in Proposition. Then $\overline{C} \subset \overline{Gr^\lambda} \cap \partial S_\lambda^+ = \overline{Gr^\lambda} \cap \psi^{-1}(H_\lambda)$

$\Rightarrow \exists D_1 \in \operatorname{Irr}(\overline{Gr^\lambda} \cap \psi^{-1}(H_\lambda))$ containing $\overline{C}$ and $\dim(D_1) = \langle \rho, 2\lambda \rangle - 1$, and $D_1 = \coprod_{w_0\lambda \leq \nu < \lambda} D_1 \cap S_\nu^+$ locally closed
$\Rightarrow \exists \nu_1$ s.t. $C_1 := D_1 \cap S_{\nu_1}^+$ open dense in D_1, $\nu_1 \geq \mu$ else $\overline{C} \subset \overline{Gr^\lambda} \cap \partial S_\mu^+$
$\Rightarrow C_1 \in \operatorname{Irr}(\overline{Gr^\lambda} \cap S_{\nu_1}^+)$ of dim. $\langle \rho, 2\lambda \rangle - 1$ s.t. $\overline{C} \subset \overline{C_1} \Rightarrow \mu < \nu_1$ if $d > 1$ ($C_1 = \overline{C_1} \cap S_{\nu_1}$, $C = \overline{C} \cap S_\mu$)
Repeat argument to find $\nu_1, \dots, \nu_d$ s.t. $\mu \leq \nu_d < \nu_{d-1} < \dots < \nu_1 < \lambda$ and $C_i \in \operatorname{Irr}(\overline{Gr^\lambda} \cap S_{\nu_i})$ s.t. $\overline{C} \subset \overline{C_i}$
and $\dim(C_i) = \langle \rho, 2\lambda \rangle - i$. Then $\langle \rho, \mu \rangle \leq \langle \rho, \lambda \rangle - d$ i.e. $d \leq \langle \rho, \lambda \rangle - \langle \rho, \mu \rangle \Rightarrow \dim C \geq \langle \rho, \lambda+\mu \rangle$ ∎

Corollary: $\lambda \in X_*^+(T)$, $X \subset \overline{Gr^\lambda}$ closed T-invariant subvariety. Then
$$\dim X \le \max_{\substack{\mu \in X_*(T) \\ L_\mu \in X}} \langle \rho, \lambda + \mu \rangle$$

Proof: $X \cap S_\mu \neq \emptyset$ iff $L_\mu \in X \Rightarrow X \subset \bigcup_{\substack{\mu \in X_*(T) \\ L_\mu \in X}} S_\mu \Rightarrow X \subset \bigcup_{\substack{\mu \in X_*(T) \\ L_\mu \in X}} (\overline{Gr^\lambda} \cap S_\mu)$

Convolution Product

We have $G(\mathcal{O}) \curvearrowright G(\mathcal{K}) \times Gr$ by $k \cdot (g, [h]) = (gk^{-1}, [kh])$.
Let $[g, h]$ be orbit of $(g, [h])$.

$$Gr \times Gr \xleftarrow{p} G(\mathcal{K}) \times Gr \xrightarrow{q} G(\mathcal{K}) \times^{G(\mathcal{O})} Gr \xrightarrow{m} Gr$$
$$([g],[h]) \longleftarrow (g,[h]) \longmapsto [g,h] \longmapsto [gh]$$

For $\mathcal{F}, \mathcal{G} \in D^b_{c, G(\mathcal{O})}(Gr, k)$, $\exists!\ \mathcal{F} \tilde{\boxtimes} \mathcal{G} \in D^b(G(\mathcal{K}) \times^{G(\mathcal{O})} Gr, k)$

s.t. $p^*(\mathcal{F} \boxtimes \mathcal{G}) = q^*(\mathcal{F} \tilde{\boxtimes} \mathcal{G})$ ($G(\mathcal{O})$ action is free, q^* gives equivalence)

Define $\mathcal{F} * \mathcal{G} := m_*(\mathcal{F} \tilde{\boxtimes} \mathcal{G})$

Proposition: $\mathcal{F}, \mathcal{G} \in Perv_{G(\mathcal{O})}(Gr, k) \Rightarrow \mathcal{F} * \mathcal{G} \in Perv_{G(\mathcal{O})}(Gr, k)$

Proof: Note that m is locally trivial ($G_{\mathcal{O}}$-equivariance), and $\mathcal{F} \tilde{\boxtimes} \mathcal{G}$ is perverse so it suffices to show m_* is stratified semismall.
The stratification on Gr is $\{Gr^\lambda\}_{\lambda \in X_*^+(T)}$, and the stratification for $G(\mathcal{K}) \times^{G(\mathcal{O})}$ is $\{Gr^\lambda \tilde{\times} Gr^\mu\}_{\lambda, \mu \in X_*^+}$ where $Gr^\lambda \tilde{\times} Gr^\mu := q(p^{-1}(Gr^\lambda \times Gr^\mu))$.
As $\dim Gr_\lambda = \langle 2\rho, \lambda \rangle$, then $\dim Gr^\lambda \tilde{\times} Gr^\mu = \langle 2\rho, \lambda + \mu \rangle$ so need
$$\dim (m^{-1}(x) \cap Gr^\lambda \tilde{\times} Gr^\mu) \le \langle \rho, \lambda + \mu - \nu \rangle \text{ for } x \in Gr^\nu$$
As m is $G(\mathcal{O})$-equivariant, can take $x = L_{w_0 \nu}$.

Let $X = m^{-1}(x) \cap \overline{Gr^\lambda \tilde{\times} Gr^\mu}$. Consider $\pi(X)$, where $\pi: G(\mathcal{K}) \times^{G(\mathcal{O})} Gr \to Gr$ is quotient of $G(\mathcal{K})$.

- $\pi|_X$ proper $\Rightarrow \pi(X)$ closed subvariety of $\overline{Gr^\lambda}$
- $X \to \pi(X)$ bijection $\Rightarrow$ suffices to show $\dim \pi(X) \le \langle \rho, \lambda + \mu - \nu \rangle$
- $\pi(X)$ meets finitely many S_φ^+ (as $S_\varphi^+ \cap Gr^\lambda \neq \emptyset$ iff $L_\varphi \in \overline{Gr^\lambda}$) so suffices to show $\dim(S_\varphi^+ \cap \pi(X)) \le \langle \rho, \lambda + \mu - \nu \rangle$ when $S_\varphi^+ \cap \pi(X) \neq \emptyset$.
- $x = L_{w_0 \nu}$ fixed by $T \curvearrowright Gr \Rightarrow X$ and $\pi(X)$ stable under T

- $\pi(X)$ closed, S_φ^+ attracting cell, $S_\varphi^+ \cap \pi(X) \neq \phi \Rightarrow L_\varphi \in \pi(X)$

- $\pi^{-1}(L_\varphi) = [t^\varphi, t^{w_0\nu-\varphi} G(\mathcal{O})] \in G(\mathcal{K}) \times^{G(\mathcal{O})} Gr$ $\qquad (t^\varphi, L_\psi) \in X = m^{-1}(x) \cap \overline{Gr^\lambda \tilde{\times} Gr^\mu}$

- Let $\psi = w_0\nu - \varphi$. Therefore, $S_\varphi^+ \cap \pi(X) \neq \phi \Rightarrow \exists \psi$ s.t. $\varphi + \psi = w_0\nu$ and $L_\psi \in \overline{Gr_\mu}$.

Thus, if $S_\varphi^+ \cap \pi(X) \neq \phi$, then

$$\begin{aligned}
\dim(S_\varphi^+ \cap \pi(X)) &\leq \dim(S_\varphi^+ \cap \overline{Gr^\lambda}) \\
&= \dim(S_\varphi^+ \cap Gr^\lambda) \\
&\leq \dim(S_\varphi^+ \cap Gr^\lambda) + \dim(S_\psi^+ \cap Gr^\mu) \\
&= \langle \rho, \lambda + \varphi + \mu + \psi \rangle \\
&= \langle \rho, \lambda + \mu + w_0\nu \rangle \\
&= \langle \rho, \lambda + \mu - \nu \rangle
\end{aligned}$$

Properties of Convolution (see Stefan's talk):
Let $\mathcal{F}, \mathcal{G}, \mathcal{H} \in D^b_{G(\mathcal{O})}(Gr, k)$. Then
1) $IC_0 * \mathcal{F} \simeq \mathcal{F} * IC_0 \simeq \mathcal{F}$
2) $(\mathcal{F} * \mathcal{G}) * \mathcal{H} \simeq \mathcal{F} * (\mathcal{G} * \mathcal{H})$
3) $(\mathbb{D}\mathcal{F}) * (\mathbb{D}\mathcal{G}) \simeq \mathbb{D}(\mathcal{F} * \mathcal{G})$

Fusion Product

Consider the following ind-schemes:

- $Gr_{\mathbb{A}^1} = \{(\mathcal{F}, \nu, x) : x \in \mathbb{A}^1, \mathcal{F}$ G-bundle on $\mathbb{A}^1$, ν trivialization of $\mathcal{F}$ on $\mathbb{A}^1 \setminus \{x\}\} \simeq Gr \times \mathbb{A}^1$

- $Gr_{\mathbb{A}^2} = \{(\mathcal{F}, \nu, x_1, x_2) : (x_1, x_2) \in \mathbb{A}^2$, $\mathcal{F}$ G-bundle on $\mathbb{A}^1$, ν trivialization of $\mathcal{F}$ on $\mathbb{A}^1 \setminus \{x_1, x_2\}\}$

We have a map $\pi : Gr_{\mathbb{A}^2} \to \mathbb{A}^2$ where $\pi^{-1}(x_1, x_2) \simeq \begin{cases} Gr & \text{if } x_1 = x_2 \\ Gr \times Gr & \text{if } x_1 \neq x_2 \end{cases}$
(Known as Beilinson-Drinfeld Grassmannian)

- $\widetilde{Gr_{\mathbb{A}^1} \times Gr_{\mathbb{A}^1}} = \left\{ (\mathcal{F}_1, \nu_1, \mu_1, \mathcal{F}_2, \nu_2, x_1, x_2) : \begin{array}{l} (x_1, x_2) \in \mathbb{A}^2, \mathcal{F}_1, \mathcal{F}_2 \text{ G-bundles on } \mathbb{A}^1, \\ \nu_i \text{ trivialization of } \mathcal{F}_i \text{ on } \mathbb{A}^1 \setminus \{x_i\}, \\ \mu_1 \text{ trivialization of } \mathcal{F}_1 \text{ near } x_2 \end{array} \right\}$

- $Gr_{\mathbb{A}^1} \tilde{\times} Gr_{\mathbb{A}^1} = \left\{ (\mathcal{F}_1, \mathcal{F}, \nu_1, \eta, x_1, x_2) : \begin{array}{l} (x_1, x_2) \in \mathbb{A}^2, \mathcal{F}_1, \mathcal{F} \text{ G-bundles on } \mathbb{A}^1, \\ \nu_1 \text{ trivialization of } \mathcal{F}_1 \text{ on } \mathbb{A}^1 \setminus \{x_1\}, \\ \eta : \mathcal{F}_1|_{\mathbb{A}^1 \setminus \{x_2\}} \xrightarrow{\sim} \mathcal{F}|_{\mathbb{A}^1 \setminus \{x_2\}} \text{ isomorphism} \end{array} \right\}$

Consider the diagram over $\mathbb{A}^2$:

$$\begin{array}{ccccccc}
Gr_{\mathbb{A}^1} \times Gr_{\mathbb{A}^1} & \xleftarrow{p} & \widetilde{Gr_{\mathbb{A}^1} \times Gr_{\mathbb{A}^1}} & \xrightarrow{q} & Gr_{\mathbb{A}^1} \tilde{\times} Gr_{\mathbb{A}^1} & \xrightarrow{m} & Gr_{\mathbb{A}^2} \\
((\mathcal{F}_1, \nu_1, x_1), (\mathcal{F}_2, \nu_2, x_2)) & \longleftarrow & (\mathcal{F}_1, \nu_1, \mu_1, \mathcal{F}_2, \nu_2, x_1, x_2) & \mapsto & (\mathcal{F}_1, \mathcal{F}, \nu_1, \eta, x_1, x_2) & \mapsto & (\mathcal{F}, \eta \circ \nu_1, x_1, x_2)
\end{array}$$

where for q, $\mathcal{F}$ is obtained by gluing $\mathcal{F}_1|_{\mathbb{A}^1 \setminus \{x_2\}}$ and $\mathcal{F}_2|_{D_{x_2}}$ along $\mathcal{F}_1|_{D^\times_{x_2}} \xleftarrow[\mu_1]{\sim} G \times D^\times_{x_2} \xrightarrow[\nu_2]{\sim} \mathcal{F}_2|_{D^\times_{x_2}}$

The idea is that over $(x,x)\in\mathbb{A}^2$, the diagram is the usual convolution diagram while over $(x,y)\in\mathbb{A}^2$ with $x\neq y$, the diagram is

$$Gr\times Gr \xleftarrow{p} G(\mathcal{K})\times Gr \xrightarrow{p} Gr\times Gr \xrightarrow{id} Gr\times Gr.$$

Let $\mathcal{F},\mathcal{G}\in P_{G_{\mathbb{A}^1}(\mathcal{O})}(Gr_{\mathbb{A}^1},k)$. As before, $\exists!\ \mathcal{F}\tilde{\boxtimes}\mathcal{G}$ s.t. $q^*(\mathcal{F}\tilde{\boxtimes}\mathcal{G})=p^*(\mathcal{F}\boxtimes\mathcal{G})$.
Define $\mathcal{F}*_{\mathbb{A}^1}\mathcal{G}:=m_*(\mathcal{F}\tilde{\boxtimes}\mathcal{G})\in D^b_c(Gr_{\mathbb{A}^2},k)$

Let $\tau: Gr_{\mathbb{A}^1}\to Gr$ be projection, $\tau^\circ:=\tau^*[1]\cong\tau^![-1]$ (shift to preserve perverseness)
Let $i: Gr_{\mathbb{A}^1}\hookrightarrow Gr_{\mathbb{A}^2}$ be inclusion of the diagonal. Set $i^\circ:=i^*[-1]$, $i^\bullet:=i^![1]$.

Lemma: For $\mathcal{F}_1,\mathcal{F}_2\in P_{G(\mathcal{O})}(Gr,k)$,

$$i^\circ(\tau^\circ(\mathcal{F}_1)*_{\mathbb{A}^1}\tau^\circ(\mathcal{F}_2))\cong\tau^\circ(\mathcal{F}_1*\mathcal{F}_2)\cong i^\bullet(\tau^\circ(\mathcal{F}_1)*_{\mathbb{A}^1}\tau^\circ(\mathcal{F}_2))$$

Proof: m proper so using base change, suffices to prove

$$(i')^*(\tau^\circ\mathcal{F}_1\tilde{\boxtimes}\tau^\circ\mathcal{F}_2)\cong(\tau')^\circ(\mathcal{F}_1\tilde{\boxtimes}\mathcal{F}_2)[1],\quad (i')^!(\tau^\circ\mathcal{F}_1\tilde{\boxtimes}\tau^\circ\mathcal{F}_2)\cong(\tau')^\circ(\mathcal{F}_1\tilde{\boxtimes}\mathcal{F}_2)[-1]$$

where $i':(G(\mathcal{K})\times^{G(\mathcal{O})}Gr)\times\Delta_{\mathbb{A}^1}\hookrightarrow Gr_{\mathbb{A}^1}\tilde{\times}Gr_{\mathbb{A}^1}$, $\tau':(G(\mathcal{K})\times^{G(\mathcal{O})}Gr)\times\Delta_{\mathbb{A}^1}\twoheadrightarrow G(\mathcal{K})\times^{G(\mathcal{O})}Gr$

i.e.

$$\begin{array}{ccc} G(\mathcal{K})\times^{G(\mathcal{O})}Gr\times\Delta_{\mathbb{A}^1} & \xrightarrow{m\times id} & Gr\times\Delta_{\mathbb{A}^1} \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i} \\ Gr_{\mathbb{A}^1}\tilde{\times}Gr_{\mathbb{A}^1} & \xrightarrow{m} & Gr_{\mathbb{A}^2} \end{array}
\quad\text{and}\quad
\begin{array}{ccc} G(\mathcal{K})\times^{G(\mathcal{O})}Gr\times\Delta_{\mathbb{A}^1} & \xrightarrow{\tau'} & G(\mathcal{K})\times^{G(\mathcal{O})}Gr \\ \downarrow{\scriptstyle m\times id} & & \downarrow{\scriptstyle m} \\ Gr\times\mathbb{A}^1 & \xrightarrow{\tau} & Gr \end{array}$$

and $(\tau')^\circ=(\tau')^*[1]\cong(\tau')^![1]$. ∎

Let $U=\mathbb{A}^2\setminus\Delta_{\mathbb{A}^1}$, $j:(Gr_{\mathbb{A}^1}\times Gr_{\mathbb{A}^1})|_U\cong Gr_{\mathbb{A}^2}|_U\hookrightarrow Gr_{\mathbb{A}^2}$. $\left(j_{!*}K=\mathrm{Im}({}^pH^0 j_!K\to{}^pH^0(j_*K))\right)$

Lemma: For $\mathcal{F}_1,\mathcal{F}_2\in P_{G(\mathcal{O})}(Gr,k)$, we have

$$j_{!*}((\tau^\circ\mathcal{F}_1\boxtimes\tau^\circ\mathcal{F}_2)|_U)\cong(\tau^\circ\mathcal{F}_1)*_{\mathbb{A}^1}(\tau^\circ\mathcal{F}_2).$$

In particular, $*_{\mathbb{A}^1}$ is a functor to $P_{G_{\mathbb{A}^1}(\mathcal{O})}(Gr_{\mathbb{A}^1},k)$.

Sketch: Over U, $(\tau^\circ\mathcal{F}_1)*_{\mathbb{A}^1}(\tau^\circ\mathcal{F}_2)=(\tau^\circ\mathcal{F}_1)\tilde{\boxtimes}(\tau^\circ\mathcal{F}_2)=(\tau^\circ\mathcal{F}_1)\boxtimes(\tau^\circ\mathcal{F}_2)$.
By Beilinson-Bernstein-Deligne, suffices to show that

$$i^*((\tau^\circ\mathcal{F}_1)*_{\mathbb{A}^1}(\tau^\circ\mathcal{F}_2))\in{}^pD^{\le -1}\quad\text{and}\quad i^!((\tau^\circ\mathcal{F}_1)*_{\mathbb{A}^1}(\tau^\circ\mathcal{F}_2))\in{}^pD^{\ge 1}.$$

By above Lemma, $i^*((\tau^\circ\mathcal{F}_1)*_{\mathbb{A}^1}(\tau^\circ\mathcal{F}_2))\cong\tau^\circ(\mathcal{F}_1*\mathcal{F}_2)[1]\in{}^pD^{\le-1}$ as $\mathcal{F}_1*\mathcal{F}_2$ perverse.
Other condition is similar, using other isomorphism in above Lemma. ∎

Combining Lemmas, we have $\forall\mathcal{F}_1,\mathcal{F}_2\in P_{G(\mathcal{O})}(Gr,k)$, $\tau^\circ(\mathcal{F}_1*\mathcal{F}_2)\cong i^\circ j_{!*}((\tau^\circ\mathcal{F}_1\boxtimes\tau^\circ\mathcal{F}_2)|_U)$
Let $s: Gr_{\mathbb{A}^2}\to Gr_{\mathbb{A}^2}$ be automorphism swapping x_1 and x_2. Then $(s\circ i)=i$ and the induced automorphism s_U of $(Gr_{\mathbb{A}^1}\times Gr_{\mathbb{A}^1})|_U$ swaps the two factors.

$$\Rightarrow\ \tau^\circ(\mathcal{F}_1*\mathcal{F}_2)\cong i^\circ j_{!*}((\tau^\circ\mathcal{F}_1\boxtimes\tau^\circ\mathcal{F}_2)|_U)\cong i^\circ s^* j_{!*}((\tau^\circ\mathcal{F}_1\boxtimes\tau^\circ\mathcal{F}_2)|_U)$$
$$\cong i^\circ j_{!*}(s_U)^*((\tau^\circ\mathcal{F}_1\boxtimes\tau^\circ\mathcal{F}_2)|_U)\cong i^\circ j_{!*}((\tau^\circ\mathcal{F}_2)\boxtimes\tau^\circ\mathcal{F}_1)|_U)$$
$$\cong\tau^\circ(\mathcal{F}_2*\mathcal{F}_1).$$

Restrict to a point to get $\mathcal{F}_1*\mathcal{F}_2\xrightarrow{\sim}\mathcal{F}_2*\mathcal{F}_1$.

Product of MV Cycles Examples

We will fix $x_1 = 0$ and allow the other to vary, so work with

$$Gr_{\mathbb{A}} = \{(\mathcal{F}, \nu, s) : s \in \mathbb{A}, \mathcal{F}\ G\text{-bundle on } \mathbb{A}, \nu \text{ trivialization of } \mathcal{F} \text{ over } \mathbb{A}^1 \setminus \{0,s\}\}$$

Look at fusion product on level of varieties i.e. irreducible components of $Gr^{\lambda} \cap S_{\mu}^{-}$, using lattice model.

L_1 0-lattice i.e. $L_1 \sim [g] \in Gr$ where $g \in G(\mathbb{C}[t, t^{-1}])$
L_2 s-lattice i.e. $L_2 \sim [g] \in Gr$ where $g \in G(\mathbb{C}[t, (t-s)^{-1}])$
Think of $(L_1, L_2) \in Gr_{\mathbb{A}}|_U$ where $U = \mathbb{A} \setminus \{0\}$.
Then $(L_1, L_2) = L$ where

$$L \underset{\mathbb{C}[t]}{\otimes} \mathbb{C}[[t]] = L_1 \underset{\mathbb{C}[t]}{\otimes} \mathbb{C}[[t]] \quad \text{and} \quad L \underset{\mathbb{C}[t]}{\otimes} \mathbb{C}[[t-s]] = L_2 \underset{\mathbb{C}[t]}{\otimes} \mathbb{C}[[t-s]]$$

Then take $L_0 = \lim_{s \to 0} L$.

Then if Z_1, Z_2 MV cycles with generic points L_1, L_2, then an irreducible component of $Z_1 * Z_2$ has generic point L_0.

A_2 Example: $L_1 = \langle te_1, e_2 \rangle$, $L_2 = \langle e_1, (t-s)e_2 \rangle$
so L_1, L_2 correspond to the varieties $L_{(1,0)}, L_{(0,1)}$.
Let $L = \langle te_1, (t-s)e_2 \rangle$.
Note that $(t-s)$ is invertible in $\mathbb{C}[[t]]$ and t is invertible in $\mathbb{C}[[t-s]]$
$\Rightarrow L = (L_1, L_2)$
$L_0 = \lim_{s \to 0} L = \langle te_1, te_2 \rangle \rightsquigarrow L_{(1,1)}$ as desired.

In matrices, $L_1 \rightsquigarrow \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}$, $L_2 \rightsquigarrow \begin{bmatrix} 1 & 0 \\ 0 & t-s \end{bmatrix}$, $L \rightsquigarrow \begin{bmatrix} t & 0 \\ 0 & t-s \end{bmatrix}$, $L_0 \rightsquigarrow \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}$.

A_2 Example: $L_1 = \langle e_1, e_2 + \frac{1}{s} t^{-1} e_1 \rangle$, $L_2 = \langle e_1, e_2 + \frac{1}{s}(t-s)^{-1} e_1 \rangle$
so L_1, L_2 correspond to $Gr^{\alpha} \cap S_0^- \cong \mathbb{P}^1$.
Let $L = \langle e_1, e_2 + t^{-1}(t-s)^{-1} e_1 \rangle$. Note: $t^{-1}(t-s)^{-1} = \frac{1}{s}(t^{-1} - (t-s)^{-1})$
We have $(e_2 + t^{-1}(t-s)^{-1} e_1) \otimes (t-s) = e_2 \otimes (t-s) + \frac{1}{s} t^{-1} e_1 \otimes (t-s) + \frac{1}{s} e_1 \otimes 1$

$$\equiv (e_2 + \tfrac{1}{s} t^{-1} e_1) \otimes (t-s) \quad \text{mod } L \otimes \mathbb{C}[[t]]$$
$$\equiv (e_2 + \tfrac{1}{s} t^{-1} e_1) \otimes 1 \quad \text{mod } L \otimes \mathbb{C}[[t]]$$

$\Rightarrow L \otimes \mathbb{C}[[t]] = L_1 \otimes \mathbb{C}[[t]]$. Similarly, $L \otimes \mathbb{C}[[t-s]] = L_2 \otimes \mathbb{C}[[t-s]]$.
$L_0 = \lim_{s \to 0} L = \langle e_1, e_2 + t^{-2} e_1 \rangle \rightsquigarrow Gr^{2\alpha} \cap S_0^- = 2^{nd}$ Hirzebruch surface.

Weight Functors and Fiber Functor

Proposition: For $\mathcal{F} \in P_{G(\mathcal{O})}(Gr, k)$, $\mu \in X_*(T)$, $k \in \mathbb{Z}$, $i : S_\mu^- \hookrightarrow Gr$, we have

$$H^k(Gr, i^! \mathcal{F}) =: H^k_{S_\mu^-}(Gr, \mathcal{F}) \simeq H^k_c(S_\mu^+, \mathcal{F})$$

and both terms vanish if $k \neq \langle 2\rho, \mu \rangle$.

Sketch: For $\lambda \in X_*^+(T)$, $\mathcal{F}$ perverse $\Rightarrow$ $\mathcal{F}|_{Gr^\lambda} \in D^{\leq -\langle 2\rho, \lambda\rangle}(Gr^\lambda, k)$.
From dimension estimates, $H^k_c(Gr^\lambda \cap S_\mu^+; k) = 0$ for $k > \langle 2\rho, \lambda + \mu \rangle$.

Grothendieck's dévissage $\Rightarrow$ $H^k_c(Gr^\lambda \cap S_\mu^+, \mathcal{F}) = 0$ for $k > \langle 2\rho, \mu \rangle$.
Filter support of $\mathcal{F}$ by the $\overline{Gr^\lambda}$ $\Rightarrow$ $H^k_c(S_\mu, \mathcal{F}) = 0$ for $k > \langle 2\rho, \mu\rangle$.
Analogous dual argument shows $H^k_{S_\mu^-}(Gr, \mathcal{F}) = 0$ for $k < \langle 2\rho, \mu \rangle$.
Braden's hyperbolic localization theorem gives $H^k_{S_\mu^-}(Gr, \mathcal{F}) \cong H^k_c(S_\mu, \mathcal{F})$. $\square$

[Induct on # of $G(\mathcal{O})$ orbits in $\operatorname{supp} \mathcal{F}$. Let $Gr^\lambda \subset \operatorname{supp}\mathcal{F}$ open, $Z = \operatorname{supp}\mathcal{F} \setminus Gr^\lambda$. $i : Z \hookrightarrow Gr$.
Consider $j_{\lambda !}(\mathcal{F}|_{Gr^\lambda}) \to \mathcal{F} \to i_* i^* \mathcal{F} \to$
Induction $\Rightarrow$ $H^k_c(S_\mu^+, i_* i^* \mathcal{F}) = 0$.
$H^k_c(S_\mu^+, j_{\lambda!}(\mathcal{F}|_{Gr^\lambda})) \overset{\text{proper base change}}{\cong} H^k_c(S_\mu^+ \cap Gr^\lambda, \mathcal{F}|\ldots)$ $= 0$ & $k > \langle 2\rho, \mu\rangle$]

Define $F_\mu : P_{G(\mathcal{O})}(Gr_G, k) \longrightarrow \mathrm{Vect}_k$ weight functors

$$\mathcal{F} \longmapsto H^{\langle 2\rho, \mu\rangle}_{S_\mu^-}(Gr, \mathcal{F}) \simeq H_c^{\langle 2\rho, \mu \rangle}(S_\mu^+, \mathcal{F})$$

Note: As $P_{G(\mathcal{O})}(Gr, k)$ is semisimple, this functor is exact.

Define $F : P_{G(\mathcal{O})}(Gr, k) \longrightarrow \mathrm{Vect}_k$ Hypercohomology functor

$$\mathcal{F} \longmapsto H^\bullet(Gr, \mathcal{F})$$

Theorem: 1) $F \cong \bigoplus_{\mu \in X_*(T)} F_\mu$
2) F is exact and faithful.

Sketch: 1) Let $\mathcal{F} \in P_{G(\mathcal{O})}(Gr, k)$. We will show $H^k(Gr, \mathcal{F}) \simeq \bigoplus_{\substack{\mu \in X_*(T) \\ \langle 2\rho, \mu\rangle = k}} F_\mu(\mathcal{F})$ for $k \in \mathbb{Z}$.
WLOG, $\mathcal{F}$ indecomposable with connected support. For $n \in \frac{1}{2}\mathbb{Z}$, set $Z_n = \coprod_{\substack{\mu \in X_*(T) \\ \langle \rho, \mu\rangle = n}} S_\mu^-$.
$\Rightarrow$ $\bigcup_{n \in \mathbb{Z}} Z_n$ and $\bigcup_{n \in \frac{1}{2} + \mathbb{Z}} Z_n$ are unions of connected components of Gr.
WLOG, $\operatorname{supp} \mathcal{F} \subset \bigcup_{n \in \mathbb{Z}} Z_n$. Give $Z_n \subset Gr$ subspace topology so $Z_n = \coprod S_\mu^-$ topologically

$$\Rightarrow H^k_{Z_n}(Gr, \mathcal{F}) = \begin{cases} 0 & \text{if } k \neq 2n \\ \bigoplus_{\langle \rho, \mu\rangle = n} F_\mu(\mathcal{F}) & \text{if } k = 2n \end{cases}$$

As $\overline{S_\mu^-} = \coprod_{\nu \geq \mu} S_\nu^-$, we have $\overline{Z_n} = Z_n \amalg Z_{n+1} \amalg Z_{n+2} \amalg \cdots = Z_n \amalg \overline{Z_{n+1}}$. Consider $\overline{Z_{n+1}} \overset{i}{\hookrightarrow} \overline{Z_n} \overset{j}{\hookleftarrow} Z_n$
Let $\mathcal{F}_n =$ corestriction of A to $\overline{Z_n}$. Applying $H^\bullet(\overline{Z_n}, -)$ to $i_* i^! \mathcal{F}_n \to \mathcal{F}_n \to j_* j^! \mathcal{F}_n$, get

$$\cdots \to H^k_{\overline{Z_{n+1}}}(Gr, \mathcal{F}) \to H^k_{\overline{Z_n}}(Gr, \mathcal{F}) \to H^k_{Z_n}(Gr, \mathcal{F}) \to H^{k+1}_{\overline{Z_{n+1}}}(Gr, \mathcal{F}) \to \cdots$$

For $n \gg 0$, $\operatorname{supp}\mathcal{F}$ is disjoint from $\overline{Z_n}$ as $\operatorname{supp}\mathcal{F}$ compact
$\Rightarrow$ $H^\bullet_{\overline{Z_n}}(Gr, \mathcal{F}) = 0$ for $n \gg 0$.

Using decreasing induction on n gives $H^k_{Z_n}(Gr, \mathcal{F}) = 0$ if k odd or $n > \frac{k}{2}$,

$$\begin{tikzcd} H^k_{Z_{k/2}}(Gr, \mathcal{F}) \arrow[r, "\sim"] \arrow[d, "\wr"] & H^k_{Z_n}(Gr, \mathcal{F}) \\ H^k_{Z_{k/2}}(Gr, \mathcal{F}) & \end{tikzcd} \quad \text{if } k \text{ even and } n \le \frac{k}{2}$$

Then take n small so that $\operatorname{supp} \mathcal{F} \subset \overline{Z_n}$.

2) As $P_{G(\mathcal{O})}(Gr, k)$ is semisimple, exactness is automatic.
For faithfulness, let $\mathcal{F} \neq 0$ perverse sheaf. Then $\operatorname{supp} \mathcal{F}$ is a finite union of Gr^λ.
Let Gr^λ be open in $\operatorname{supp} \mathcal{F}$.
$\Rightarrow \mathcal{F}|_{Gr^\lambda} \simeq \underline{k}[\dim Gr^\lambda]$ as Gr^λ simply connected.
$\Rightarrow S^+_{w_0\lambda} \cap \operatorname{supp} \mathcal{F} = S^+_{w_0\lambda} \cap Gr^\lambda$

Similarly, $S^-_\lambda \cap \operatorname{supp} \mathcal{F} = S^-_\lambda \cap Gr^\lambda = \{L_\lambda\}$.
$F_\lambda(\mathcal{F}) \cong H^{\langle 2\rho, \lambda \rangle}(S^-_\lambda, (i_\lambda)^! \mathcal{F})$
$\cong H^{\langle 2\rho, \lambda \rangle}(S^-_\lambda \cap Gr^\lambda, i^!(\mathcal{F}|_{Gr^\lambda})) \simeq k$

We have $\dim(S^+_{w_0\lambda} \cap Gr^\lambda) = 0$. In fact, $S^+_{w_0\lambda} \cap Gr^\lambda = \{L_{w_0\lambda}\}$
$\Rightarrow F_{w_0\lambda}(\mathcal{F}) = H_c^{-\langle 2\rho, \lambda \rangle}(\mathcal{F}|_{S^+_{w_0\lambda}}) \cong H^{-\langle 2\rho, \lambda \rangle}(\mathcal{F}|_{S^+_{w_0\lambda} \cap Gr_\lambda}) \cong k$. ▯

Proposition: For $\mathcal{F}_1, \mathcal{F}_2 \in P_{G(\mathcal{O})}(Gr, k)$, $F(\mathcal{F}_1 * \mathcal{F}_2) = F(\mathcal{F}_1) \otimes_k F(\mathcal{F}_2)$.

Idea: Let $\mathcal{F}_1, \mathcal{F}_2 \in P_{G(\mathcal{O})}(Gr, k)$. Set $\mathcal{F} = (\tau^\circ \mathcal{F}_1) *_{\mathbb{A}^1} (\tau^\circ \mathcal{F}_2)$. Let $\pi : Gr_{\mathbb{A}^2} \to \mathbb{A}^2$ be projection. For each $k \in \mathbb{Z}$,

- kth cohomology sheaf of $(\pi_* \mathcal{F})|_{\Delta \mathbb{A}^1}[-2]$ is locally constant on $\Delta \mathbb{A}^1$ with stalk $H^k(Gr, \mathcal{F}_1 * \mathcal{F}_2)$
- kth cohomology sheaf of $(\pi_* \mathcal{F})_U[-2]$ is locally constant on U with stalk $H^k(Gr \times Gr, \mathcal{F}_1 \boxtimes \mathcal{F}_2) \simeq \bigoplus_{i+j=k} H^i(Gr, \mathcal{F}_1) \otimes H^j(Gr, \mathcal{F}_2)$ (Künneth formula)

Then $H^k(Gr, \mathcal{F}_1 * \mathcal{F}_2) \simeq \bigoplus_{i+j=k} H^i(Gr, \mathcal{F}_1) \otimes H^j(Gr, \mathcal{F}_2)$

if we know $\mathcal{H}^{k-2}(\pi_* \mathcal{F})$ is locally constant on $\mathbb{A}^2$. ▯

Note that F factors through $\mathrm{Vect}_k(X_*(T))$:

$$\begin{tikzcd} P_{G(\mathcal{O})}(Gr, k) \arrow[r, "\bigoplus_\mu F_\mu"] \arrow[dr, "F"'] & \mathrm{Vect}_k(X_*(T)) \arrow[d, "\text{forget}"] \\ & \mathrm{Vect}_k \end{tikzcd}$$

Proposition: For $\mathcal{F}_1, \mathcal{F}_2 \in P_{G(\mathcal{O})}(Gr, k)$, $\mu \in X_*(T)$,
$$F_\mu(\mathcal{F}_1 * \mathcal{F}_2) = \bigoplus_{\mu_1+\mu_2=\mu} F_{\mu_1}(\mathcal{F}_1) \otimes_k F_{\mu_2}(\mathcal{F}_2)$$

Proof: See [BR].

Lemma: $\mathcal{F} \in P_{G(\mathcal{O})}(Gr, k)$. We have $\dim F(\mathcal{F}) = 1$ iff $\mathcal{F} \cong IC_\lambda$ for some λ with $\langle 2\rho, \lambda \rangle = 0$.

Proof: If $\mathcal{F} \cong IC_\lambda$ with $0 = \langle 2\rho, \lambda \rangle = \dim Gr^\lambda$ so $Gr^\lambda = \overline{Gr^\lambda} = pt$ and $\mathcal{F}$ is a skyscraper sheaf on this point. Then $H^\bullet(Gr, \mathcal{F}) = k$.
Conversely, suppose $\dim H^\bullet(Gr, \mathcal{F}) = 1$. The support of $\mathcal{F}$ must be closure of a single $G(\mathcal{O})$-orbit, else $F_{\mu_1}(\mathcal{F}) \neq 0 \neq F_{\mu_2}(\mathcal{F})$ for $\mu_1 \neq \mu_2$.

Let $\overline{Gr^\lambda}$ be this orbit. If $\langle 2\rho, \lambda \rangle > 0$, then $\exists$ root α s.t. $\langle \alpha, \lambda \rangle \neq 0$ so $\lambda \neq w_0\lambda$.
Then in proof of F faithful, can show that $F_\lambda(\mathcal{F}) \neq 0 \neq F_{w_0\lambda}(\mathcal{F})$.
$\Rightarrow \langle 2\rho, \lambda \rangle = 0$.
$\Rightarrow$ support of $\mathcal{F}$ is a 0-dim. $G(\mathcal{O})$-orbit so $\mathcal{F}$ skyscraper on $Gr_\lambda = \overline{Gr_\lambda}$
$\Rightarrow \mathcal{F} \cong IC_\lambda$.

Note that if $\langle 2\rho, \lambda \rangle = 0$, then $-\lambda$ is dominant.

The Dual Group

By Tannakian Formalism, we have

$$\begin{tikzcd} P_{G(\mathcal{O})}(Gr, k) \arrow[rr, "\sim", "S"'] \arrow[dr, "F"'] & & Rep_k(\tilde{G}) \arrow[dl, "\omega"] \\ & Vect_k & \end{tikzcd}$$

for some group $\tilde{G}$ over k.

Lemma: $\tilde{G}$ is algebraic.

Proof: Let $\lambda_1, \ldots, \lambda_n$ generate $X_*^+(T)$. For $\lambda = \sum k_i \lambda_i$, $k_i \in \mathbb{Z}_{\geq 0}$, the sheaf IC_λ is a direct summand of $\underbrace{IC_{\lambda_1} * \cdots * IC_{\lambda_1}}_{k_1 \text{ copies}} * \cdots * \underbrace{IC_{\lambda_n} * \cdots * IC_{\lambda_n}}_{k_n \text{ copies}}$
$\Rightarrow IC_{\lambda_1} \oplus \cdots \oplus IC_{\lambda_n}$ is a tensor generator for $P_{G(\mathcal{O})}(Gr, k) \simeq Rep_k(\tilde{G})$
$\Rightarrow \tilde{G}$ is algebraic.

Lemma: $\tilde{G}$ is connected.

Proof: If $\lambda \in X_*^+(T)$ nonzero, then $IC_{m\lambda}$ nonisomorphic for $m \in \mathbb{Z}_{\geq 0}$ (different supports).
$\Rightarrow$ for $\mathcal{F} \in P_{G(\mathcal{O})}(Gr, k)$ nonzero, full subcategory formed by subquotients of objects $\mathcal{F}^{\otimes n}$ cannot be stable under $*$. The same true for $Rep_k(\tilde{G})$ $\Rightarrow \tilde{G}$ connected.

Lemma: $\tilde{G}$ is reductive.

Proof: If $\bar{k}$ is algebraic closure of k, then from Tannakian formalism,

$$P_{G(\mathcal{O})}(Gr, \bar{k}) \simeq Rep_{\bar{k}}(Spec(\bar{k}) \times_{Spec(k)} \tilde{G})$$

and $Rep_{G(\mathcal{O})}(Gr, \bar{k})$ is semisimple. Thus $\tilde{G}$ is reductive. ∎

We will now construct a split maximal torus in $\tilde{G}$.

Let $T^\vee$ be unique split k-torus s.t. $X^*(T^\vee) = X_*(T)$.
$\Rightarrow Vect_k(X_*(T)) \simeq Rep_k(T^\vee)$

$\bigoplus_\mu F_\mu$ induces a functor $F_{T^\vee}: Rep_k(\tilde{G}) \to Rep_k(T^\vee)$ so we have

$$\begin{tikzcd} P_{G(\mathcal{O})}(Gr, k) \arrow[r, "\oplus F_\mu"] \arrow[d, "S"'] \arrow[dr, phantom, "\circlearrowleft"] & Vect_k(X_*(T)) \arrow[d, phantom, "\wr"] \\ Rep_k(\tilde{G}) \arrow[r, "F_{T^\vee}"'] & Rep_k(T^\vee) \end{tikzcd}$$

$\Rightarrow F_{T^\vee}$ induced by a unique morphism $\varphi: T^\vee \to \tilde{G}$.

Each $\lambda \in X^*(T^\vee)$ appears in at least one $F_{T^\vee}(IC_\mu)$ (e.g. μ = dominant in $W\lambda$)
$\Rightarrow \varphi$ is an embedding of a closed subgroup so $T^\vee$ viewed as a split torus in $\tilde{G}$.
[Jantzen]
$F_{T^\vee}$ gives morphism $T^\vee_{\mathbb{Q}} \to Spec(\mathbb{Q} \otimes_{\mathbb{Z}} K^0(Rep_{\bar{k}}(\tilde{G}_{\bar{k}}))) \simeq T^\vee_{\mathbb{Q}}/W$
$\Rightarrow$ rank $\tilde{G} = \dim T^\vee \Rightarrow T^\vee$ is a maximal torus of $\tilde{G}$.

Remains to identify root datum of $(\tilde{G}, T^\vee)$. WLOG, $k = \bar{k}$.

Consider $2\rho \in X^*(T)$. Then $\exists$ Borel $\tilde{B} \subset \tilde{G}$ s.t. $T^\vee \subset \tilde{B}$ and 2ρ is a dominant coweight for choice of positive roots of $\tilde{G}$ given by the $T^\vee$-wts in Lie algebra of $\tilde{B}$.

Lemma: For such a choice of $\tilde{B}$, the dominant wts for $T^\vee$ are exactly the dominant coweights $X_*^+(T)$ of T.

Proof: For $\lambda \in X_*^+(T)$, let $V = S(IC_\lambda)$ be the simple $\tilde{G}$-module corresponding to IC_λ.
Maximal value for $\langle 2\rho, \mu \rangle$ for μ a wt of V is when $\mu = \lambda$ and only for this wt. Then λ dominant for $T^\vee \subset \tilde{B} \subset \tilde{G}$ and is highest wt for V.

Conversely, let $\mu \in X^*(T^\vee)$ be dominant for $T^\vee \subset \tilde{B} \subset \tilde{G}$. Let V be simple of h.wt. μ.
Then $V = (IC_\lambda)$ $\exists!$ $\lambda \in X_*(T)^+$, and by above, $\lambda = \mu$. Thus μ dominant for $T \subset B \subset G$. ∎

Note: $\tilde{B}$ is uniquely determined.

Let $\Delta(\tilde{G}, T^\vee)$ be the root system
$\Delta_+(\tilde{G}, \tilde{B}, T^\vee)$ be the positive roots
$\Delta_s(\tilde{G}, \tilde{B}, T^\vee)$ be the simple roots
Similar notation for G.

We have $\{\mathbb{Q}_+ \cdot \alpha : \alpha \in \Delta_s^\vee(\tilde{G}, \tilde{B}, T^\vee)\} = \{\mathbb{Q}_+ \cdot \beta : \beta \in \Delta_s(G, B, T)\}$

Lemma: $\Delta_s(\tilde{G}, \tilde{B}, T^\vee) = \Delta_s(G, B, T)$ as subsets of $X_*(T) = X^*(T^\vee)$.

Proof: Let $G^\vee$ be Langlands dual to G. Then $T^\vee$ is a maximal torus of $G^\vee$.
Choose positive roots of $(G^\vee, T^\vee)$ as positive coroots of $T \subset B \subset G$.
$\Rightarrow$ dominant weights of $(G^\vee, T^\vee)$ are $X_*^+(T)$.

Let $\lambda \in X_*^+(T)$. Consider simple $G^\vee$-module $V_\lambda(G^\vee)$ of h.wt. λ and simple $\tilde{G}$-module $V_\lambda(\tilde{G}) = S(IC_\lambda)$ of h.wt. λ. These two have the same weights.

Observe that $\{\lambda - \mu : \lambda \in X_*^+(T), \mu \text{ a wt of } V_\lambda(G^\vee)\}$ is the $\mathbb{N}$-span of positive roots of $(G^\vee, T^\vee)$.
$\Rightarrow$ also $\mathbb{N}$-span of positive roots of $(\tilde{G}, T^\vee)$.
$\Rightarrow$ simple roots of $\tilde{G}$ are the simple roots of $G^\vee$. ∎

Theorem: $\tilde{G}$ is Langlands dual to G.
Proof: We have $X^*(T^\vee)$ dual to $X^*(T)$. Need to show roots and coroots of $\tilde{G}$ with canonical bijection between them coincide with coroots and roots of G with their canonical bijection.

Let $\alpha \in \Delta_s(G, B, T)$. Then $\alpha^\vee \in \Delta_s(\tilde{G}, \tilde{B}, T^\vee)$. The coroot $\tilde{\alpha}$ of $\tilde{G}$ associated to this root is $\mathbb{Q}_+$-proportional to a simple root of $T \subset B \subset G$.
We have
- $\langle \tilde{\alpha}, \alpha^\vee \rangle = 2$
- $\langle \tilde{\alpha}, \beta^\vee \rangle \leq 0$ for $\beta^\vee \in \Delta_s(\tilde{G}, \tilde{B}, T^\vee) \setminus \{\alpha^\vee\}$

$\Rightarrow \tilde{\alpha} = \alpha$.
Thus $\Delta_s(G, B, T) = \Delta_s^\vee(\tilde{G}, \tilde{B}, T^\vee)$. We also have $\Delta_s(\tilde{G}, \tilde{B}, T^\vee) = \Delta_s^\vee(G, B, T)$ and the bijections between simple roots and coroots are the same.
$\Rightarrow$ can identify Weyl groups of G and $G^\vee$ and extend equalities to all roots and coroots. ∎

Concluding Remarks

The simples IC_λ correspond to the simples of h. wt. λ $L(\lambda)$.
($F_\lambda(IC_\lambda) \neq 0$ and wts of $S(IC_\lambda)$ are $\leq \lambda$ as $F_\mu(\mathcal{F}) \neq 0 \Rightarrow L_\mu \in \operatorname{supp} \mathcal{F}$)

$M(\lambda)$ Weyl module of h. wt. λ characterized by
- $\exists\, M(\lambda) \twoheadrightarrow L(\lambda)$ whose kernel has wts $< \lambda$
- $\forall M$ with wts $< \lambda$, $\mathrm{Hom}(M(\lambda), M) = 0$

These are standard objects and correspond to $I_!(\lambda) := {}^pH^0(j_{\lambda !}\, \underline{k}_{Gr^\lambda}[\dim Gr^\lambda])$
where $j_\lambda : Gr^\lambda \hookrightarrow Gr$.

The costandard objects are dual Weyl modules corresponding to
$I_*(\lambda) := {}^pH^0((j_\lambda)_* \underline{k}_{Gr^\lambda}[\dim Gr^\lambda])$.

Addendum

S_μ^+ are attraction varieties:
Let $2\check{\rho} : \mathbb{G}_m \to T$ be sum of positive coroots. Acting by conjugation by $2\check{\rho}$ on $N(K)$, we have $\lim_{s\to 0} 2\check{\rho}(s) n = 1$ for all $n \in N(K)$.
$\Rightarrow$ for any $x \in S_\mu^+$, $\lim_{s \to 0} 2\check{\rho}(s) x = L_\mu$. Since L_μ are the fixed points of the $\mathbb{G}_m$-action via $2\check{\rho}$,

$$S_\mu^+ = \{x \in Gr : \lim_{s\to 0} 2\check{\rho}(s) x = L_\mu\}.$$