Ergodic Properties of Folding Maps on Spheres

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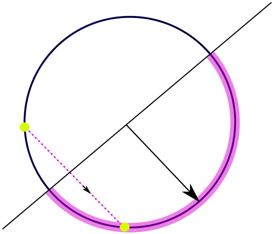
OMC 2015



Radial Symmetry

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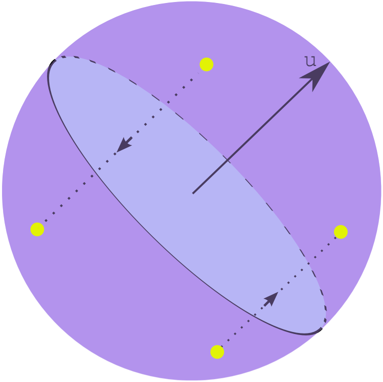
- When is a function radial to begin with? When it is invariant under rotations (representation theory & dynamics)

Definitions and Notation

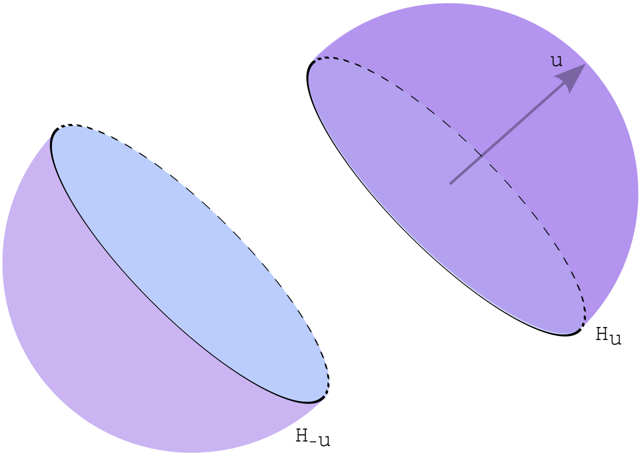
For a direction vector $u \in \mathbb{S}^{d-1}$ define

- the ***reflection*** $R_u : x \mapsto x - 2(x \cdot u)u$



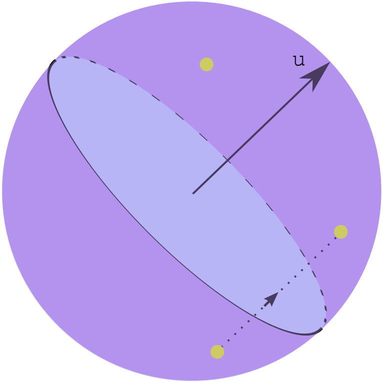
Definitions and Notation

- the ***positive half-space*** $H_u = \{x \in \mathbb{S}^{d-1} | x \cdot u > 0\}$



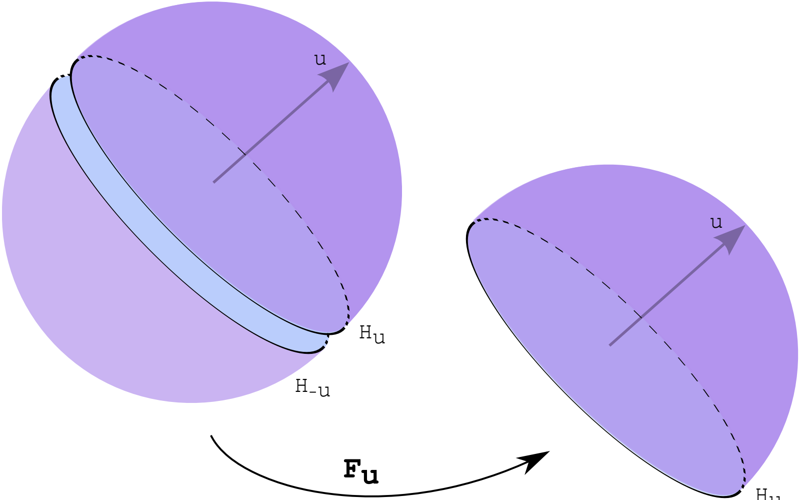
Definitions and Notation

- the ***folding map*** $F_u : x \mapsto R_u(x)$ if $x \notin H_u$ and $x \mapsto x$ otherwise



Definitions and Notation

It is a $2:1$ non-expansive piecewise isometry. It folds $\mathbb{S}^{d-1}$ onto H_u.



Two such maps do not in general commute.

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- the ***trajectory*** of x under the sequence of maps (F_{u_n}) to be the set
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We may refer to this set as the trajectory of x under the sequence of directions (u_n).

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For a subset $G \subset \mathbb{S}^{d-1}$ define

- the ***orbit*** of x under G to be the set
$$G_* x = \{F_{u_n} \cdots F_{u_1} x | n \geq 1, u_i \in G\}.$$

What partial information ensures full radial symmetry?

Let $G \subset \mathbb{S}^{d-1}$ be a set of directions. Let ϕ be a continuous function on $\mathbb{R}^d$.

- $\phi \circ R_u = \phi$ for all $u \in G \Rightarrow \phi$ is radial

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- The subgroup $\langle G \rangle$ generated by $\{R_u | u \in G\}$ is dense in $O(d)$.

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- Can we generate dense trajectories?

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Necessary conditions are

- (cover) Half-spaces got from G cover $\mathbb{S}^{d-1}$

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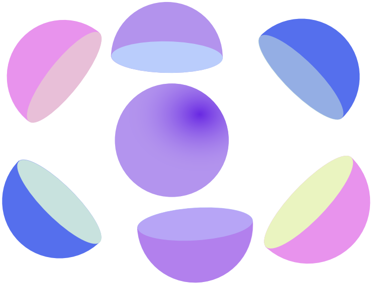

- (generate) The subgroup generated by G is dense in $O(d)$

$$\overline{\langle G \rangle} = O(d)$$

Main Results

Theorem

If $G \subset \mathbb{S}^{d-1}$ a subset of directions satisfies

(cover) *the hyperplanes H_u cover $\mathbb{S}^{d-1}$, and*

(generate) *the subgroup (of reflections!) generated by G is dense in $O(d)$*

then there exists a sequence $(u_n)_{n\geq 1}$ in G such that for all initial $x \in \mathbb{S}$, the trajectory

$$x_n = F_{u_n} x_{n-1} \qquad x_0 = x$$

is dense.

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Let μ be a probability measure on $\mathbb{S}^{d-1}$ with

(cover) $0 < \mu(H_u) < 1$ *for all* $u \in \mathbb{S}^{d-1}$

(generate) $\langle \mathrm{Supp}\mu \rangle$ *is dense in* $O(d)$.

Define a random walk by

$$X_n = F_{U_n} X_{n-1} \qquad X_0 = x$$

with (U_n) *i.i.d.* $\sim \mu$.

There is a unique invariant measure (wrt F) on $\mathbb{S}^{d-1}$ and the random walk starting at x is (a.s.) dense.

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- almost surely = with probability 1 = almost everywhere

The covering condition

Proposition

Let $G \subset \mathbb{S}^{d-1}$. TFAE

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2. *G not in any closed hemisphere*
3. *interior of convex hull of G contains origin*
   - *In particular, G must contain at least $d+1$ directions that span $\mathbb{R}^d$*
4. *For the G a subset of the support of a measure μ, $0 < \mu(H_x) < 1$ for all $x \in \mathbb{S}^{d-1}$*

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  1. lies in a lower-dimensional subgroup of $O(d)$
  2. splits into two subgroups which act on mutually orthogonal subspaces (is reducible)
  3. defines a finite Coxeter subgroup of $O(d)$

A sufficient generating condition

Suppose

1. G spans $\mathbb{R}^d$
2. not all angles between elements of G are commensurable with π, and
3. G is not the union of two non-empty orthogonal subsets

then $\langle G \rangle$ is dense in $O(d)$.

A sufficient algebraic condition

Proof.

Induct on d.

- If G satisfies conditions 1-3, then it contains a subset G' that spans a hyperplane $v^{\perp} \cong \mathbb{R}^d$ and also satisfies 1-3

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- Since $SO(d+1)$ contains no proper compact subgroup which contains a copy of $SO(d)$, $\langle S_v, S_w \rangle$ is dense

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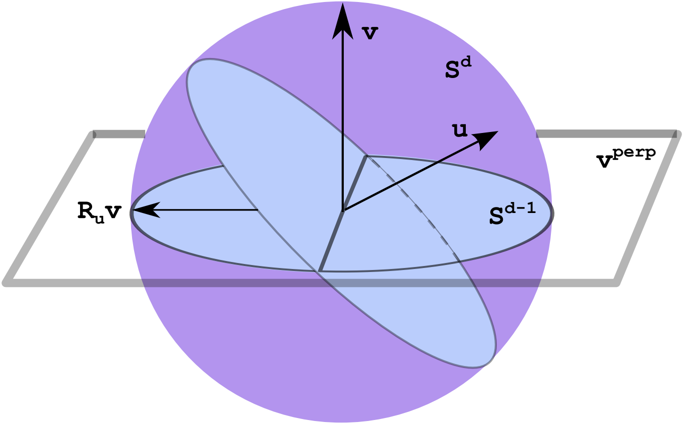
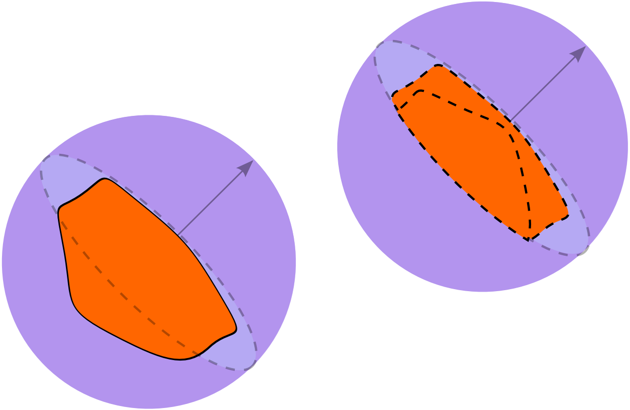


□

Some more definitions

A subset $A \subset \mathbb{S}^{d-1}$ is

- ***positively invariant*** if $F_u(A) \subset A$ for all $u \in G$



- ***almost positively invariant*** if $\sigma(A \backslash F_u(A)) = 0$ for all $u \in G$
- ***invariant*** if $R_u(A) = A$ for all $u \in G$, and
- ***almost invariant*** if $\sigma(A \Delta R_u A) = 0$ for all $u \in G$.

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Proof of Theorem 1

A is ***almost positively invariant*** $\Rightarrow$ A is ***almost invariant*** $\Rightarrow$ A is measure 0 or 1 $\Rightarrow$ $A = \emptyset$ or $\mathbb{S}^{d-1}$

Conclusion

For each point $x \in \mathbb{S}^{d-1}$ the orbit $G_* x$ is positively invariant,

$$F_u G_* x = F_u \{ F_{u_n} \cdots F_{u_1} x | n \geq 1, u_i \in G \}$$

hence dense. Density of trajectories follows by concatenating sequences gauranteed by density of the orbit.

- cover the sphere by finitely many balls $B_1, ..., B_K$ of centers $c_1, .., c_K$ and radius ϵ

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- the the sequence $R_K \cdots R_1$ connects any two points x, y

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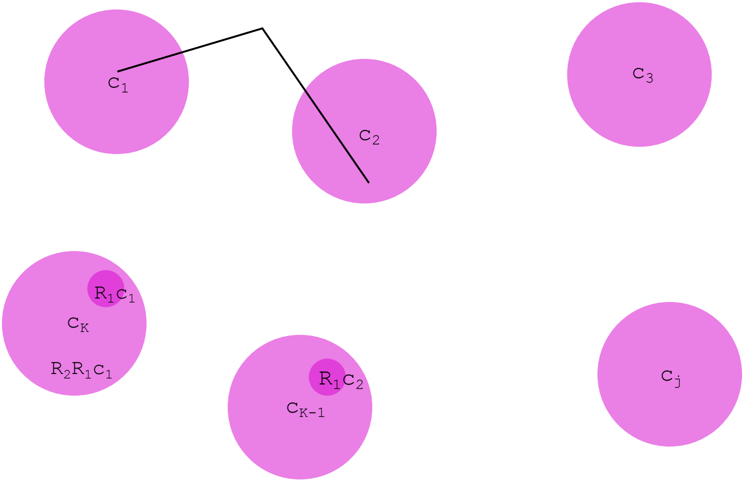

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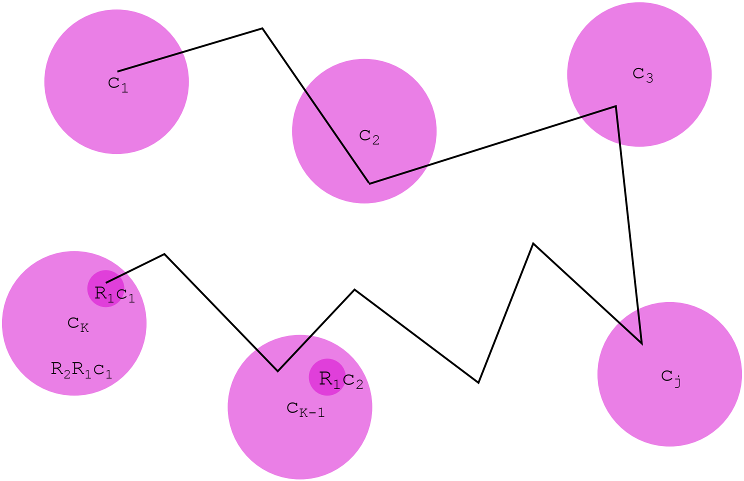

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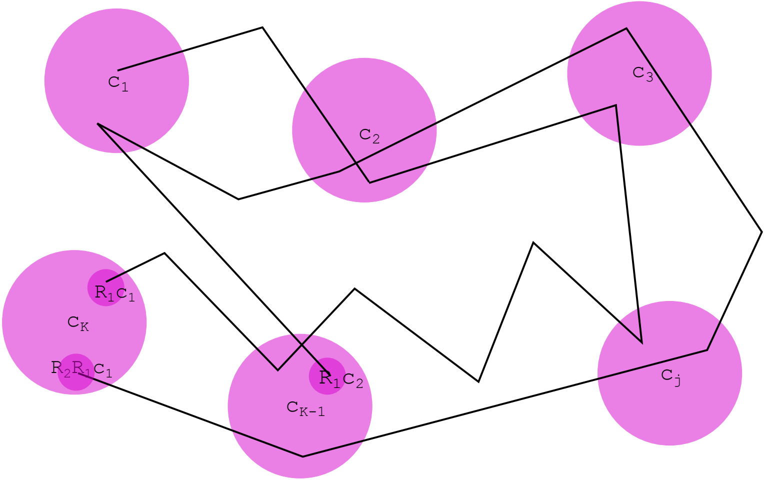

Thank you