A Hitchhiker’s Guide to the Affine Grassmannian

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Figure 1: Stratum of $Gr_{SL_2}$ (alias today’s goal)
What is... an affine Grassmannian
Fix a complex reductive group $G$.

Write $O = \mathbb{C}[[t]]$ for power series (a PID) and $K$ for Laurent series (Frac $O$).

**Definition A.** $Gr = G(K)/G(O)$. 
Fix a compact connected Lie group $U$.

**Definition B.** Write $LU$ for maps $S^1 \to U$ and $\Omega U$ for those maps sending some fixed $z_0 \in S^1$ to $1 \in U$. $LU, \Omega U$ are groups under $f \cdot g(z) = f(z)g(z)$. Call them loop groups.

$\Omega U \cong LU/U$ by

1. constructing a map $LU \to \Omega U : f \mapsto f(z_0)^{-1}f$
2. considering the “loop rotation” action $w \cdot f(z) = f(wz)$ of $S^1$ on $LU$
Fix $n \in \mathbb{Z}$.

Let $O$ be a PID and $K$ its field of fractions.

**Recall.** A lattice $L$ is a free $O$-module of the vector space $K^n$ such that $K \otimes_O L \cong K^n$ as vector spaces.

**Definition C1.** $Gr = \{L : L \cong O^n\}$ (as $O$-modules).

This set carries a natural action of $GL_n K$. We recover **Definition A** by checking that the stabilizer of a given lattice is (isomorphic to) $GL_n O$.

Henceforth $G = GL_n K$, $O = \mathbb{C}[[t]]$ and $K = \mathbb{C}((t))$ and note the other-way-map

$$[g] \in G(K)/G(O) \mapsto O^n g^{-1}$$
Fix $L \in Gr$.

Set

$$V_d(L) = \frac{t^{-d} L \cap O^n}{(tO)^n} \subseteq \frac{O^n}{(tO)^n} \cong \mathbb{C}^n$$

**Lemma.** $V_d(L)$ is increasing in $d$ from 0 to $\mathbb{C}^n$. 
\(L \in Gr\) has a basis whose elements have some least power of \(t\). Therefore multiplication by \(t^{-d}\) for \(d > 0\) has the effect of pulling \(L\) over \(O^n\).
**Corollary.** $Gr$ can be written as a union of finite dimensional schemes.

*Reason.* $Gr$ is a union of

$$Gr[a, b] = \{ t^b O^n \subseteq L \subseteq t^a O^n \} \quad (a \leq b)$$

which can be identified with closed subschemes in some $Gr(k, (b - a)n)$ since $t^a O^n / t^b O^n \cong \mathbb{C}^{(b-a)n}$.

Thus e.g. the “Bruhat decomposition” -every invertible matrix $M$ can be reduced to a unique permutation matrix $\tilde{w}$ by upward row operations, rightward column operations and scaling columns- used to produce a basis of Schubert cycles for $H^\bullet$ of finite $Gr$s can be carefully generalized to affine $Gr$s.
For a finer decomposition consider the $\mathbb{C}^\times$ action on $Gr$

$$z \in \mathbb{C}^\times : L \mapsto zL = L$$

which scales $t$ so that for $L = \text{Span}_O(v_1, \ldots, v_n)$ for $v_i = \sum v^i_{jk} e_j t^k$

$$tv_i = \sum v^i_{jk} e_j (zt)^k$$

Taking $z \to 0$ has the effect of picking off least powers of basis elements, a tuple in $\mathbb{Z}^n$ which can be interpreted as a vertex of a moment polytope or as a coweight for $G$.

$$z \cdot (3e_1 t^{-1} + 7e_3 t^5 + e_6)$$
$$= z^{-1}(3e_1 t^{-1} + 7e_3 z^6 t^5 + ze_6)$$
$$= 3e_1 t^{-1} + 7e_3 z^6 t^5 + ze_6 \to 3e_1 t^{-1}$$
For $T \subset G$ a maximal torus, $X_*(T) = \text{Hom}(\mathbb{C}^\times, T) \cong \mathbb{Z}^n$.

There is a map $X_*(T) \to \text{Gr}$ via the map $X_* \to G(K)$ defined by post-composing $\lambda : \mathbb{C}^\times \to T$ and $\text{Spec } K \to \mathbb{C}^\times$ identifying $\lambda \in X_*$ and $t^\lambda \equiv \text{diag}(t^{\lambda_1}, \ldots, t^{\lambda_n}) \in GL_n K$ or under the other-way map $L_\lambda = \text{Span}_O(e; t^{\lambda_i} : 1 \leq i \leq n)$. 
Related Facts.

- The fixed points of the $T$ action on $Gr$ are indexed by $X_*(T)$.
- The $z \to 0$ limits of the $\mathbb{C}^\times$ action on $Gr$ are indexed by $X_*(T)$.
- The $G(O)$ orbits of $Gr$ contain unique $T$-fixed points.
- Finally

$$Gr = \bigsqcup_{\lambda \in X_*} Gr^\lambda$$
Geometric Satake : ( 

\[ H^\bullet : IC_{Gr^\lambda} \in \mathcal{P}_G \mapsto V(\lambda) \in \text{Rep}_G \] 

Case \( \lambda = \omega_k \) : )

\[ H^\bullet(Gr(k, n)) \cong \bigwedge^k \mathbb{C}^n \quad \text{dim} = \binom{n}{k} \]

Schubert varieties make up the basis on the left and \( k \)-element subsets of \( n \) index a basis on the right.
**Emulating** $\omega_k$. Consider the linear map $t \cdot : K^n \to K^n$ sending $e_i t^j$ to $e_i t^{j+1}$ induced by multiplication by $t$ on $K$.

**Definition C2.** $Gr^> = \{ L \in Gr : t \cdot L \subset L \}$ sometimes called the *positive part of* $Gr$.

Fix $\lambda, \mu \in X_\ast(T)$ non-decreasing. Write $Gr^\lambda$ for the set

$$\{ L \in Gr^> : t|_{L/L_0} \text{ has jordan type } \lambda \}$$

and $S^\mu$ for the set

$$\{ L \in Gr^> : \lim_{z \to 0} z \cdot L = L_\mu \}$$

where $L_\mu = \text{Span}_O(e_1 t^{\mu_1-1} \ldots e_\nu t^{\mu_\nu-1})$ and $z \cdot$ is our $\mathbb{C}^\times$ action from before.
Fact. The set $\text{Gr}^\lambda \cap S^\mu$ has dimension equal $\dim V(\lambda)_\mu$ and its irreducible components, the so-called MV cycles, form a basis for $H^\bullet(\text{Gr}^\lambda)$ endowing it with a $X_\bullet$ grading, generalizing the case $\lambda = \omega_k$. 
There is an action on $H^\bullet(\overline{Gr^\lambda})$ by multiplication by $c(\mathcal{L})$ where $\mathcal{L}$ denotes the det bundle on $Gr$ and $c$ Chern class.

**Fact.** This action is secretly an action of $gl_n$. It decomposes as

$$c_{\mu\nu} : H^\bullet(\overline{Gr^\lambda \cap S^\mu}) \to H^\bullet(\overline{Gr^\lambda \cap S^\nu})$$

with $c_{\mu\nu}$ nonzero only if $\nu = \mu + \alpha_i$ so that letting $E_i, F_i \in gl_n$ act by the appropriate components of $c(\mathcal{L}), c(\mathcal{L})^*$ defines $H^\bullet(\overline{Gr^\lambda})$ as an irrep of $gl_n$. 
Definition D. Let $\lambda \geq \mu \in X_*$ viewed as partitions of $N$ and consider the subset of $\mathfrak{gl}_N$ defined by $\mathcal{O}_\lambda \cap \mathcal{T}_\mu$ where $\mathcal{O}_\lambda = \text{GL}_N \cdot J_\lambda$ and by example $\mathcal{T}_{(3,2,2)}$ is elements of the form

\[
\begin{bmatrix}
0 & 1 & & & & & \\
0 & 1 & & & & & \\
* & * & * & * & * & * & * \\
0 & 1 & 0 & & & & \\
* & * & * & * & * & & \\
0 & 1 & & & & & \\
* & * & * & * & * & & \\
\end{bmatrix}
\]

call it $M^\lambda_{\mu}$. 
**Fact.** The lattice POV supplies $M^\lambda_\mu \cong Gr^\lambda_\mu$ with $L \in Gr^\lambda_\mu$ being sent to the matrix of $t$

$$[t|_{L_0/L}]_B$$

in the basis

$$B = \{[e_1] \ldots [e_1 t^{\mu_1-1}], \ldots, [e_n] \ldots [e_n t^{\mu_n-1}]\}$$
Examples
Fix $G = SL_2$, $\lambda = (2, 0)$, and $\mu = (1, 1)$. 
In $Gr = G(K)/G(O)$ one defines

- $Gr_{\mu} = G_1[[t^{-1}]]t^\mu$ for $G_1 = \text{Ker}(ev_\infty : Gr \mapsto G)$
- $Gr^\lambda = G(O)t^\lambda$
- $Gr^{\lambda}_{\mu} = \overline{Gr^\lambda \cap Gr_\mu}$
**Fact.** \( K^\times \cong \mathbb{Z} \times O^\times \) or \( 0 \neq g \in K \) can be written \( t^n f \) for \( f = f_0 + hot \) and \( n \in \mathbb{Z} \).

Using this fact and the definitions, check that

\[
G(O)t^{(2,0)}G(O) \cap G_1[[t^{-1}]]t^{(1,1)}G(O)
\]

\[
= \left\{ \begin{bmatrix} t + a & b \\ c & t + d \end{bmatrix} : \det = t^2 + (a + d)t + (ad - bc) = t^2 \right\}
\]

\[
\cong \{ a + d = 0, a^2 + bc = 0 \}
\]

the 2-dimensional variety from slide 1.
On the other side $M_{(1,1)}^{(2,0)} = O_{(2,0)}$ and we check that

\[
\begin{bmatrix}
    a & b \\
    c & d
\end{bmatrix} \cdot \begin{bmatrix}
    0 & 1 \\
    0 & 0
\end{bmatrix} \begin{bmatrix}
    a & b \\
    c & d
\end{bmatrix}^{-1} = \begin{bmatrix}
    -ac & a^2 \\
    -c^2 & ac
\end{bmatrix}

= \begin{bmatrix}
    z & x \\
    -y & z
\end{bmatrix} : z^2 + xy = 0
\]
What else is the affine Grassmannian
Definition E. Trivializable bundles definition.

4.2. Global picture. Let $X$ be a curve, which in our case will always be $\mathbb{A}^1$. Let $\mathbb{A}^{(n)} = \mathbb{A}^1 \times \cdots \times \mathbb{A}^1 // \mathfrak{S}_n$ be the symmetric $n$-fold product of $\mathbb{A}^1$

Beilinson-Drinfeld Grassmannian [BD, MVi1, MVi2] is a (reduced) ind-scheme $\mathcal{G}_{\mathbb{A}^{(n)}}$ whose $\mathbb{C}$-points are described as follows:

(22) $\mathcal{G}_{\mathbb{A}^{(n)}}(\mathbb{C}) = \{(b, \mathcal{V}, t) \mid t : \mathcal{V}_{X-E} \to (X \times V)|_{X-E} \text{ is an isomorphism }\}$,

where $b = (b_1, \ldots, b_n) \in \mathbb{A}^{(n)}$, $E = \{b_1, \ldots, b_n\} \subseteq \mathbb{A}^1$, $\mathcal{V}$ is a vector bundle of rank $m$, and $t$ is the trivialization of $\mathcal{V}$ off $E$. The pairs $(\mathcal{V}, t)$ are considered up to an isomorphism.
Thank you for listening