Quiver Representations and Quiver Varieties

Anne Dranowski

Mathematics
UNIVERSITY OF TORONTO

Abstract

Canonical bases of representations constructed from the geometry of homogeneous spaces involve many choices, so in order to compare seemingly different constructions, representation theorists use combinatorics to parametrize coarse invariants, like dimensions of kernels of Chevalley generators. But, no matter how much we filter our representations, dimension generally cannot distinguish basis vectors themselves. We show that for quiver varieties and preprojective algebra modules—two sources of combinatorially equivalent representations—a finer invariant analogous to the Euler class does the trick.

Table 1. Dramatis personae

$G, G^\vee$	simple simply-connected complex algebraic group, its Langlands dual group
T, N, B	usual data associated to G
$Q = (I, E)$	Dynkin quiver of G
$L(\lambda)$	irreducible G-representation of highest weight λ
$\mathcal{T}(\lambda)_\mu$	semistandard Young tableaux of shape λ and content μ in type A
$\mathcal{G}r$	affine Grassmannian $G^\vee((t))/G^\vee[[t]]$ of $G^\vee$
$\alpha_i, \alpha_i^\vee$	simple roots, coroots indexed by I
t^μ, L_μ	points in $G^\vee(\mathcal{K}), \mathcal{G}r$ defined by G-weight μ
$\mathcal{G}r_\mu$	orbit of L_μ under $G_1[[t^{-1}]] = \mathrm{Ker}(G[[t^{-1}]] \to G : t \mapsto \infty)$
$\mathcal{G}r^\lambda, S^\mu_\pm$	spherical- and $\frac{\infty}{2}$-orbits in $\mathcal{G}r$ defined by G-weights λ, μ
Z, v_Z	typical MV cycle, its basis vector in $\mathbb{C}[N]$
$\mathbb{O}_\lambda$	conjugacy class of nilpotent matrices defined by G-weight λ
$\mathbb{T}_\mu$	Mirković–Vybornov slice of matrices defined by G-weight μ
$\Lambda(\nu)$	Lusztig's nilpotent variety of ν-dimensional representations of Q
M, u_M	the typical generic module for a component of $\Lambda(\nu)$, its basis vector in $\mathbb{C}[N]$

Setting the scene

Our story rests on the following foundation.

1. The irreducible components of $\overline{\mathcal{G}r^\lambda} \cap \overline{S^\mu_-}$, alias **MV cycles of coweight** (λ, μ), index a basis of the μ-weight space of $L(\lambda)$. [MV07]
2. Their moment polytopes (for an action of T on $\mathcal{G}r$), alias **MV polytopes of coweight** (λ, μ), have explicit combinatorial definitions in terms of certain tuples of integers called Lusztig data. [Kam10]
3. In type A, the slice $\overline{\mathcal{G}r^\lambda} \cap \mathcal{G}r_\mu$ is isomorphic to an affine quiver variety $\mathcal{M}_0 \cong \overline{\mathbb{O}}_\lambda \cap \mathbb{T}_\mu$ for (a representation of) an A_{μ_1} quiver. [MV19]
4. In simply laced type, the irreducible components of $\Lambda(\nu)$, alias **MV modules**, index a basis of $\mathbb{C}[N]_{-\nu}$ via $\mathbb{C}[N] \cong \mathcal{U}\mathfrak{n}^*$ for $\mathfrak{n} = \mathrm{Lie}(N)$ [Lus00].
5. Their **Harder–Narasimhan polytopes** also have explicit combinatorial definitions in terms of Lusztig data, and for a given datum $\mathbf{n}_\bullet$ the associated HN and MV polytopes are equal. [BK12].

Main results

Theorem A. [Dra19] *The irreducible components of $\mathcal{M}_0 \cap \mathfrak{n}$ are indexed by $\mathcal{T}(\lambda)_\mu$ via an explicit map taking a tableau τ to the matrix variety X_τ defined by the open rank conditions*

$$\{A \in \mathcal{M}_0 \cap \mathfrak{n} : A|_{\mathbb{C}^{\mu_1 + \cdots + \mu_i}} \in \mathbb{O}_{\mathrm{shape}(\tau^{(i)})} \text{ for } 1 \le i \le m\}$$

Theorem B. [Dra19] *The Mirković–Vybornov isomorphism $\mathcal{M}_0 \cong \overline{\mathcal{G}r^\lambda} \cap \mathcal{G}r_\mu$ restricts to to an isomorphism $\mathcal{M}_0 \cap \mathfrak{n}$ and $\overline{\mathcal{G}r^\lambda} \cap S^\mu_-$ which we call ψ. Moreover, the projective closure Z_τ of $\overline{\psi(X_\tau)}$ in $\overline{\mathcal{G}r^\lambda} \cap \overline{S^\mu_-}$ is an MV cycle with Lusztig datum $\mathbf{n}_\bullet$ determined by τ).*

Theorem C. [BKK19, Appendix] *Let $\mathbf{n}_\bullet$ be the Lusztig datum determined by the tableau*

$$\tau = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 3 & 3 \\ \hline 2 & 2 & 5 & 5 \\ \hline 4 & 4 \\ \cline{1-2} 6 & 6 \\ \cline{1-2} \end{array}$$

The associated MV cycle-MV module pair (Z, M) defines distinct vectors $v_Z \neq u_M$ in $\mathbb{C}[N]$. Thus, outside of small rank, polytope correspondence does not imply vector equality.

Towards Theorem C

Definition. *We say that Z and M are* **extra-compatible** *if for all $n \in \mathbb{N}$ and all weights μ we have*

$$\dim \Gamma(Z, \mathcal{O}(n))_\mu = \chi(\{0 \subseteq N_1 \subseteq \cdots \subseteq N_n \subseteq M : \sum \dim N_k = -\mu\})$$

Example 3. Taking $n = 1$ gives $\dim \Gamma(Z, \mathcal{O})_\mu = \chi(\{N \subseteq M : \dim N = -\mu\})$ which can be viewed as an upgrade of equality of polytopes.

Question. Are equivariant invariants of Z and the structure of a general point M related? How is this connected to the relationship between the basis vectors v_Z and u_M?

Evidence for A_5 extra-compatibility. Let

$$\tau = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1} \end{array}$$

Then X_τ is the vanishing locus of the ideal

$$(a_5, a_{10}, a_1 a_6 + a_2 a_8, a_7 a_8 - a_6 a_9, a_1 a_7 + a_2 a_9)$$

in $\mathbb{C}[\mathfrak{n}]$ where this time a_i are the matrix entries of a nilpotent upper triangular matrix. The ideal is defined by the following generic rank conditions.

$$\begin{bmatrix} 0 & a_1 \\ 0 & 0 \end{bmatrix} \in \mathbb{O}_{(2)} \quad \begin{bmatrix} 0 & a_1 & a_2 \\ 0 & 0 & a_5 \\ 0 & 0 & 0 \end{bmatrix} \in \mathbb{O}_{(2,1)} \quad \begin{bmatrix} 0 & a_1 & a_2 & a_3 \\ 0 & 0 & a_5 & a_6 \\ 0 & 0 & 0 & a_8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{O}_{(2,2)} \quad A \in \mathbb{O}_\lambda$$

Its multidegree in $\mathbb{T}_\mu \cap \mathfrak{n}$ is

$$(\alpha_1\alpha_2 + \alpha_2^2 + \alpha_2\alpha_3)\alpha_4^2 + (\alpha_1\alpha_2^2 + \alpha_2^3 + \alpha_2\alpha_3^2 + 2\,(\alpha_1\alpha_2 + \alpha_2^2)\alpha_3)\alpha_4\,.$$

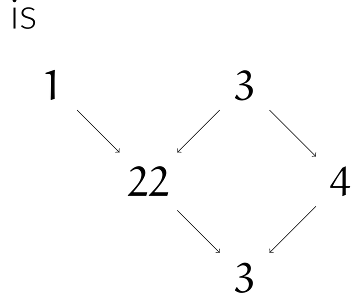
The projective closure of the associated MV cycle Z_τ is given by the homogenization of the ideal

$$(b_9 - b_3 b_6 + b_4 b_5, b_{10} - b_2 b_6, b_{11} - b_2 b_8, b_{12} - b_2 b_5, b_{13} - b_2 b_7, b_{14} - b_3 b_8 + b_4 b_7, b_{15} - b_1 b_8, b_{16} - b_1 b_7, b_1 b_5 + b_2 b_7, b_6 b_7 - b_5 b_8, b_1 b_6 + b_2 b_8)$$

where b_i are Plücker coordinates (minors of an augmented matrix $\tilde{A}$) on an ordinary Grassmannian into which $\mathcal{G}r^\lambda$ embeds. Using `Macaulay2` we obtain

$$\dim \Gamma(Z, \mathcal{O}(n)) = \frac{(n+1)^2(n+2)^2(n+3)(5n+12)}{144}\,.$$

The general module M_τ defined by τ is



with the maps chosen such that $\mathrm{Ker}(M_2 \to M_3)$, $\mathrm{Im}(M_3 \to M_2)$ and $\mathrm{Im}(M_1 \to M_2)$ are all distinct.

The composition series

$$F_i(M_\tau) = \{0 = M^0_\tau \subseteq M^1_\tau \subseteq \cdots \subseteq M^m_\tau = M_\tau : M^k_\tau / M^{k-1}_\tau \cong S_{i_k} \text{ for all } k\}$$

are used to construct the so-called flag function $\bar{D}(M) := \sum_i \chi(F_i(M))\,\bar{D}_i$. It is this analogue of the Euler class that enables us to compare vectors in $\mathbb{C}[N]$. In this example, the sequences

$(3,4,2,3,2,1)$ $(3,2,4,3,2,1)$ $(2,3,4,2,3,1)$ $(2,3,2,1,4,3)$
$(2,3,2,4,3,1)$ $(2,3,4,2,1,3)$ $(2,3,2,4,1,3)$ $(3,2,1,2,4,3)$
$(3,4,2,1,2,3)$ $(3,2,4,1,2,3)$ $(3,2,1,4,2,3)$

define $F_i(M) \cong \mathrm{pt}$, so $\chi(F_i(M)) = 1$, while the sequences

$(3,4,2,2,3,1)$ $(3,2,4,2,3,1)$ $(3,2,2,4,3,1)$ $(3,4,2,2,1,3)$
$(3,2,4,2,1,3)$ $(3,2,2,4,1,3)$ $(3,2,2,1,4,3)$

define $F_i(M) \cong \mathbb{P}^1$, so $\chi(F_i(M)) = 2$. For all other values of i, $F_i(M) = \varnothing$.

The flag function is a rational function, but we can use the multidegree $p(\mu)$ of $\mathbb{T}_\mu$ to clear the denominator. By direct computation we obtain that $\bar{D}(M_\tau)p(\mu) = \mathrm{mdeg}_{\mathbb{T}_\mu \cap \mathfrak{n}}(Z_\tau)$. In fact, we can show (using the Auslander-Buchsbaum formula and `Macaulay2`) that in this case that Z_τ is projectively normal, and that as a consequence (M_τ, Z_τ) are extra-compatible!

Appendix: Equivariant invariants of MV cycles

Most Λ-modules M are not extra-compatibly paired with any MV cycle; for example if $G = SL_3$ and M is the sum of the two simple Λ-modules, then the rhombus $\mathrm{Pol}(M)$ is the union of two MV polytopes, each a triangle. However, for any Λ-module M, we expect that there will be a corresponding coherent sheaf on the affine Grassmannian, supported on a union of MV cycles. The Euler characteristic of the "quiver grassmannian"

$$\mathrm{Gr}_\mu(M[t]/t^n) := \{N \subseteq M \otimes \mathbb{C}[t]/t^n : N \text{ is a } \Lambda \otimes \mathbb{C}[t]\text{-submodule of } \dim N = \mu\}$$

coincides with that of the flag variety $F_{n,\mu}(M)$.

Conjecture. [BKK19] *For any preprojective algebra module M of dimension vector ν, there exists a coherent sheaf $\mathcal{F}_M$ supported on $\overline{S^0_+ \cap S^{-\nu}_-}$ such that*

$$\Gamma(\mathrm{Gr}, \mathcal{F}_M \otimes \mathcal{O}(n)) \cong H^\bullet(\mathbb{G}(M[t]/t^n))$$

as $T^\vee$-representations. For example, if Z and M are extra-compatible, then we can take $\mathcal{F}_M = \mathcal{O}_Z$.

From $f \in \mathbb{C}[N]$, $e_{\mathbf{i}} = e_{i_1} \cdots e_{i_p} \in \mathcal{U}\mathfrak{n}$, and $D_{\mathbf{i}} = (\pi_{\mathbf{i}})_*(\text{Lebesgue measure})$ with $\pi(e_p) = \alpha_1 + \cdots + \alpha_p$ we construct the piecewise polynomial measures on $\mathfrak{t}^*_{\mathbb{R}}$

$$D(f) = \sum_{\mathbf{i}} \langle e_{\mathbf{i}}, f \rangle D_{\mathbf{i}}$$

Using the fact that the exponential functions $e^\beta(x) = e^{\langle \beta, x \rangle}$ form a basis for the meromorphic functions on $\mathfrak{t}_{\mathbb{C}}$ we can consider the Fourier transforms $\hat{f}(\beta) = \int_{\mathfrak{t}_{\mathbb{C}}} f(x) e^{-\beta}(x)\,dx$. In particular, the Fourier transform of a distribution on $\mathfrak{t}^*$ is the meromorphic function $\mathrm{FT}(\mu)(x) = \int_{\mathfrak{t}^*_{\mathbb{R}}} e^\beta(x)\,d\mu$ for $x \in \mathfrak{t}_{\mathbb{C}}$. For $f \in \mathbb{C}[N]_{-\nu}$ define $\bar{D}(f)$ to be $\widehat{\mathrm{FT}(D(f))}(-\nu)$ aka the coefficient of $e^{-\nu}$ in the expansion of $\mathrm{FT}(D(f))$.

Theorem. *The equivariant multiplicity of an MV cycle Z of coweight $(0, -\nu)$ at the point $L_{-\nu}$ is equal to $\bar{D}(v_Z) = \widehat{\mathrm{FT}(D(v_Z))}(-\nu)$ is equal to the $p(\mu)$-normalized multidegree.*

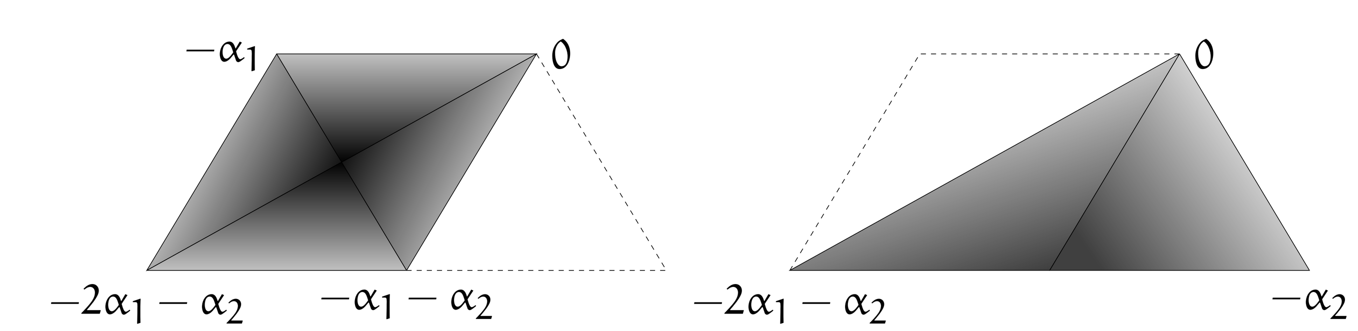


Figure 1. The $SL(3)$ examples of $D_{\mathbf{i}}$ for $\mathbf{i} = (1, 2, 1)$ and $(2, 1, 1)$, vertices labeled by their positions, with the shading to suggest the (piecewise-linear function times Lebesgue) measure.

Rules for computing multidegrees. Let $T = (\mathbb{C}^\times)^m$ be a torus, and suppose $(X \subset W)$ is a pair of linear T-reps, with X a T-invariant closed subscheme.

- If $Z = W = \{0\}$, then $\mathrm{mdeg}_W X = 1$.
- If Z has top-dimensional components Z_i, then $\mathrm{mdeg}_W X = \sum_i \mathrm{mdeg}_W X_i$.
- If Z is irreducible and H is a T-invariant hyperplane in W, then if $Z \not\subset H$, then $\mathrm{mdeg}_W Z = \mathrm{mdeg}_H(Z \cap H)$; if $Z \subset H$, then $\mathrm{mdeg}_W Z = (\mathrm{mdeg}_H Z) \cdot \nu$, where ν is the weight of T on W/H.

References

[BK12] Baumann and Kamnitzer. Preprojective algebras and MV polytopes. *Representation Theory of the American Mathematical Society*, 16(5):152–188, 2012.

[BKK19] Baumann, Kamnitzer and Knutson. The Mirković–Vilonen basis and Duistermaat–Heckman measures with an Appendix by A. Dranowski, J. Kamnitzer and C. Morton-Ferguson. *arXiv preprint arXiv:1905.08460*, 2019.

[Dra19] Dranowski. Generalized orbital varieties for Mirković–Vybornov slices as affinizations of Mirković–Vilonen cycles. *arXiv preprint arXiv:1905.08174*, 2019.

[Kam10] Joel Kamnitzer. Mirković–Vilonen cycles and polytopes. *Annals of Mathematics*, 171(1):245–294, 2010.

[Lus00] George Lusztig. Semicanonical bases arising from enveloping algebras. *Advances in Mathematics*, 151(2):129–139, 2000.

[MV07] Ivan Mirković and Kari Vilonen. Geometric Langlands duality and representations of algebraic groups over commutative rings. *Annals of mathematics*, pages 95–143, 2007.

[MV19] Ivan Mirkovic and Maxim Vybornov. Comparison of quiver varieties, loop Grassmannians and nilpotent cones in type A with an Appendix by Vasily Krylov. *arXiv preprint arXiv:1905.01810*, 2019.