Let X a smooth variety $/\mathbb{C}$ of dim$^{=}$ d_X. $(D_X, f^{\prime}D_Y)$-Mod

Recall: $f: X \to Y$, $f_*: D^r(X) \to D^r(Y)$

$M \longmapsto f_\bullet(M \otimes_{D_X} f^{\prime} D_Y)$

Def^c: $H^\bullet_{dR}(X,-) := D(X) \longrightarrow \mathrm{Vect}$, $\quad D(X) \xrightarrow{dR} Sh(X) \xrightarrow{\pi_\bullet} \mathrm{Vect}$ gen by π_*. $\pi: X \to pt$

$$H^\bullet_{dR}(M) = \pi_\bullet(M \otimes_{D_X} \mathcal{O}_X) = \pi_\bullet(\omega_X \otimes_{D_X} M^\ell)$$

$$\begin{array}{ccccccc} & & \cdots & -1 & & 0 & \\ D_X \otimes_{\mathcal{O}_X} \mathrm{Sym}^\bullet(\Theta_X[1]) = & \cdots \to & & D_X \otimes \Theta_X & \xrightarrow{mult.} & D_X & \twoheadrightarrow \mathcal{O}_X \\ \Omega^\bullet_X \otimes_{\mathcal{O}_X} D_X[2n] = & \cdots \to & & \Omega^{d_X-1}_X \otimes_{\mathcal{O}_X} D_X & \xrightarrow{d} & \Omega^{d_X}_X \otimes_{\mathcal{O}_X} D_X & \twoheadrightarrow \omega_X \\ & & \cdots & -n-1 & & -n & \end{array}$$

$M \longmapsto M^{an} \otimes_{D^{an}_X} \mathcal{O}^{an}_X$.

$dR: D(X) \longrightarrow Sh(X)$.

$\mathcal{O}_X[-2n] \to$ const.
$\mathcal{O}_X[-n] \to IC$
$\mathcal{O}_X \to$ dualizing

$(\ni \omega_X)$ $\mathcal{O}_X \longmapsto \omega_X \otimes_{D_X} \mathcal{O}_X \simeq \Omega^\bullet_X[2n]$ $\boxed{\simeq \underline{\mathbb{C}}_X[2n].}$

$(D^r \ni) D_X \longmapsto \mathcal{O}_X \otimes_{D_X} D_X$

$N \otimes_{\mathcal{O}_X} D_X \longmapsto N$ for $N \in QCoh(X)$.

$\left[\overset{0}{E \otimes_{\mathcal{O}} D_X} \xrightarrow{P\cdot} \overset{1}{F \otimes_{\mathcal{O}_X} D_X}\right] \longmapsto$ Sheaf of (derived) sol^{n}s to $P\sigma = 0$, $\sigma \in E$.

$\mathrm{Hom}_D(E \otimes_{\mathcal{O}} D, F \otimes_{\mathcal{O}} D) = \mathrm{Diff}(E,F)$.

Natural Quest.: When is $dR(M) \in Sh_c(X)$?

Let $\mathrm{Conn}(X) \subset D(X)^\heartsuit$ the full subcategory on locally $\mathcal{O}_X$-free objects, i.e. VB's with flat connection.

For X smth proj, $\mathrm{Conn}(X) \overset{GAGA}{\simeq} \mathrm{Conn}(X^{an}) \simeq \mathrm{Loc}(X)$. by classical [illegible] PDE!

For X smooth, $\mathrm{Conn}^{rg}(X) \simeq \mathrm{Conn}(X^{an}) \simeq$ " "

need "regularity". subtle!

Toward building subcat. of $D(X)$ equivalent to $Sh_c(X)$.

expect: $\mathrm{Conn}^{reg}(X) \xrightarrow{\simeq} \mathrm{Loc}(X)$.

$SS(M) = X$ iff $M \in \mathrm{Conn}(X)$.

Singular Support: "How big is M over D_X?"

Recall filtration $\mathcal{O}_X \subset \mathcal{O}_X + \Theta_X \subset D_X^{\leq 2} \subset \cdots \subset D_X$.

$\sigma: D_X \xrightarrow{\sim} gr_\bullet D_X = \mathcal{O}(T^*X)$.

$(x_i, \partial_{x_i}) \longmapsto (x_i, \xi_i)$

Def^c: Let $M \in D_{coh}(X)^\heartsuit$ have "good" filtration. compatible s.t. $gr_\bullet M \in gr_\bullet D_X$-Mod

$$SS(M) = \mathrm{supp}(gr_\bullet M)^{red}$$

$\dim SS(M) \geq d_X$

E.g.:

$\mathcal{O}_X = \mathbb{C}[x,\partial_x]/(\partial_x) \rightsquigarrow SS(\mathcal{O}_X) = X$. ✓

$D_X = \mathbb{C}[x,\partial_x] \rightsquigarrow SS(D_X) = T^*X$ ✗

$\delta_x = \mathbb{C}[x,\partial_x]/(x) \rightsquigarrow SS(\delta_x) = T^*_xX$. ✓

$Z \overset{i}{\hookrightarrow} X$ $\quad i_*\mathcal{O}_X \rightsquigarrow N^*_{Z/X}$. ✓

$i_* D_X \rightsquigarrow T^*X|_Z$ ✗

Def^{n}: $M \in D(X)^{\heartsuit}$ is holonomic if $\dim SS(M) = d_X$

Let $D_{hol}(X) =$ cx's w. hol^{c} cohom. $\left(\simeq D\left((D(X)^{\heartsuit})_{hol}\right)\right)$

Th^{m}: $M \in D_{coh}(X)$ is holonomic iff there exists a stratification $X = \bigsqcup_\alpha X_\alpha$ s.t. $SS(M) \subset \bigsqcup_\alpha N^*_{X_\alpha}$, and moreover,

$$H^k(\iota^!_\alpha M) \in \mathrm{Conn}(X)[-k]$$

$\forall\, k \in \mathbb{Z}, \alpha$. (Also iff stalks of cohom sheaves are f.dim^{l})

First, $M \in D_{hol}(X)^{\heartsuit}$.

R^{mk} on Pf: Suppose WLOG $\mathrm{supp}(M) = X$. Then $\exists\; U \overset{j}{\hookrightarrow} X$ open st. $j^! M \in \mathrm{Conn}(X)$, since $SS(M) \to T^*X \searrow X$ must be generically finite, and is conical, thus generically an iso^{m}, since only fibres are $\emptyset$, pt.

Def^{n}: $M \in D_{hol}(X)$ is regular if its composition factors are $\iota_{*!} L$ for $\iota : Z \hookrightarrow X$, $L \in \mathrm{Conn}^{reg}(X)$

Th^{m}: $dR : D_{reg,hol}(X) \xrightarrow{\simeq} Sh_c(X)$,

$$\begin{tikzcd} D_{reg,hol}(X) \arrow[r, "\simeq"] & Sh_c(X) \\ D_{r,h}(X)^{\heartsuit} \arrow[u, phantom, "\cup"] \arrow[r, "\simeq"] & \mathrm{Perv}(X) \arrow[u, phantom, "\cup"] \end{tikzcd}$$

s.t. and the functors (π^*, π_*), $(\pi_!, \pi^!)$, $\mathbb{D}$, $\boxtimes$,

Rmk on Pf: On some stratification, we know $M \in D_{rh}(X)^{\heartsuit}$ is a complex of connections $M^\bullet_{X_\alpha}$ in non-neg degrees

Then $\iota^!_\alpha dR(M) = dR(\iota^!_\alpha M) = \mathrm{tot}(\Omega^\bullet_{X_\alpha} \otimes M^\bullet_{X_\alpha}[d_{X_\alpha}])$

Now, suppose $G \circlearrowright X$, $G \times X \overset{act}{\underset{\pi_X}{\rightrightarrows}} X$

For $F \in QCoh(X)$, a G-equivariant str is $\Phi : act^* F \xrightarrow{\simeq} \pi_X^* F$ in $QCoh(G \times X)$ st...

$M \in D(X)^{\heartsuit}$, a "weak" " $\Phi : act^! M \xrightarrow{\simeq} \pi_X^! M$ ($\mathcal{O} \boxtimes D$-modules)

" " " strong " " " $D(\times)$.

Given weak str, define $\mathcal{L}_{(-)} : \mathfrak{g} \to \mathrm{End}_{D(X)}(M)$.

$$\mathcal{L}_x s = \frac{d}{dt}\Big|_{t=0} \Phi^*_{e^{tx}}(s), \quad \Phi_g := \Phi|_{\{g\}\times X} : act^!_g M \xrightarrow{\simeq} M.$$

$M \in (D(X)^{\heartsuit})^G$ iff $M \in (D(X)^{\heartsuit})^{G,w}$ and $\mathcal{L}_x = \nabla_x$

$M \in D(X)^G$ by def^{c} $M \in D(X)^{G,w}$ and $\mathcal{L}_x - \nabla_x = [d, h]$ for $h : \mathfrak{g} \to \mathrm{End}^{-1}(M)$.

E.g: Let $X = pt$, G connected. $\mathrm{Rep}(G)$

$\boxed{D_c(X)^{\geq -d_X}}$

$D(pt)^{G,w} = \{(V,d) \in D(pt) = \mathrm{Vect},\ \Phi_g \in \mathrm{Aut}(V)\ \forall g \text{ st} \ldots\}$

$$\begin{tikzcd} D(pt)^G = \{(V,d),\ \Phi : G \to \mathrm{Aut}(V),\ h : \mathfrak{g} \to \mathrm{End}^{-1}(V) \text{ st. } \mathcal{L}_x = [d,h] = d\rho\} \arrow[d, "\simeq"'] & (V,d,h) \arrow[d, mapsto] \\ H^\bullet_G(pt)\text{-Mod.} & \left((V \otimes_{\mathbb{C}} \mathbb{C}[u_i])^G,\ d + u_i h_{x_i}\right) \end{tikzcd}$$

$H^\bullet_G(pt) = \mathcal{O}(\mathfrak{h}[-2]/W) = \mathbb{C}[u_i]^W$, u_i deg 2, parameters indexed by basis for $\mathfrak{h}$.

Pf: On any non-zero wt space V^λ,

$[d, h] = d\rho = \mathbb{1} \cdot \lambda$.

Thus, h provides homtpy contracting V^λ. ▨

Exercise: For $f : X \to Y$ G equiv., define $f_* : D_G(X) \to D_G(Y)$

For $\pi : X \to pt$, check $\pi_* : D_G(X) \to D_G(pt) \simeq H^\bullet_G(pt)$ is Cartan model

At level of $\mathcal{O}$-sections. $\Phi : G \to \mathrm{Aut}(\Gamma_{\mathcal{O}}(X,M))$

$\mathcal{L}_x = d\Phi : \mathfrak{g} \to \mathrm{End}(\Gamma_{\mathcal{O}}(X,M))$

For $M \in D(X)^{\heartsuit}$, M G equivariant iff $\nabla_x \in \mathrm{End}(\Gamma_{\mathcal{O}}(X,M))$ is integrable.

Note: A property, not a structure!

Moreover, the latter is how we got rep's via B.B.!.

We have

$$\Gamma : D_\lambda(G/B)^{\heartsuit} \xrightarrow{\simeq} \mathfrak{g}\text{-Mod}_\lambda^{\heartsuit}$$

induces equivalences $D_\lambda(G/B)^{\heartsuit,K} \xrightarrow{\simeq} \mathfrak{g}\text{-Mod}_\lambda^{\heartsuit,K} \underset{\text{is}}{} (\mathfrak{g},K)\text{-Mod}_\lambda^{\heartsuit}$

$\forall$ $K \subset G/B$.

Thus,

$$\tilde{\mathcal{O}}_0 = \mathfrak{g}\text{-Mod}_{(f.g.),0}^{\heartsuit,N} \xrightarrow{\simeq} D_{(coh)}(G/B)^{\heartsuit,N} \xrightarrow{\simeq} D_{(coh)}(G/B)^{\heartsuit,B}.$$

Prop$^{\underline{o}}$: Let X a smooth variety, $B \subset X$ with finitely many orbits. Then $D_{rh}(X)^{\heartsuit,B} \xhookrightarrow{\simeq} D_{coh}(X)^{\heartsuit,B}$ an equivalence.

Cor$^{\underline{e}}$: $\tilde{\mathcal{O}}_0 \to D_{coh}(G/B)^{\heartsuit,B} \simeq \mathrm{Perv}_B(G/B)$.

Irred. obj's $\longleftrightarrow \{X_w$ a B-orbit, $L \in \mathrm{Loc}^{irr}(X_w)^B\}$

Recall, we saw the BGG res. for $L(0)$ was.

$$\mathcal{O}_X \sim L(0) \overset{\|}{} w_0\cdot(-2\rho). \qquad M(w\cdot(-2\rho)).$$

$$\mathcal{O}_X \downarrow \; j_{w_0*}j_{w_0}^{!}\mathcal{O}_X \to \underbrace{\bigoplus_{l(w)=n-1} j_{w*}j_w^{!}\mathcal{O}_X} \to \cdots \to j_{e*}j_e^{!}\mathcal{O}_X.$$

More generally, fix $w \in W$; $l(w)=k$, recall BGG res. is

?.

exact here.

$$j_w^{!}j_{w*}\mathcal{O}_X \to \bigoplus_{\substack{l(y)=k-1\\ y<w}} j_{y*}j_y^{!}\mathcal{O}_X \to \bigoplus_{\substack{l(y)=k-2\\ y<w}} \text{'' ''} \to j_{e*}j_e^{!}\mathcal{O}_X$$

Cor: $L(w\cdot(-2\rho)) = ? = j_{w,!*}\mathcal{O}_{X_w} \in D_{rh}(G/B)^{\heartsuit,B}$.

The Kazhdan - Lusztig Conjecture:

$$[L(w\cdot-2\rho)] = \sum_{y\le w} (-1)^{l(w)-l(y)} P_{y,w}(1)\cdot[M(y\cdot-2\rho)]$$

For some polynomials P.

Using structure of $\mathrm{Perv}_B(G/B) \subset D_B(G/B)$, we'll determine P's, "prove" Thm!