X smooth variety $/\mathbb{C}$, dim $X = n$.

$\overset{(r)}{D^{\ell}}(X)$ = Complexes of sheaves of $\overset{\text{(right)}}{\text{left}}$ modules over D_X quasicoherent over $\mathcal{O}_X$.

$D_X = \langle \underset{\mathcal{O}_X}{\overset{f}{\wedge}}, \underset{\Theta_X}{\overset{\Theta}{\wedge}} \rangle$

Recall: $\overset{h}{\overset{\wedge}{\mathcal{O}_X}} \in D^{\ell}(X)$. $h \mapsto f\cdot h$ $h \longmapsto \Theta(h)$

turns out $\Omega^n_X \in D^r(X)$ $\omega \longmapsto f\cdot\omega$.

$\omega \longmapsto -\mathrm{Lie}_{\Theta}\omega$

$\mathrm{Lie}_{f\Theta}\omega = d\,\iota_{f\Theta}\omega = d\,\overset{\curvearrowright}{f\,\iota_{\Theta}}\omega = \mathrm{Lie}_{\Theta}(f\cdot\omega)$.

$\mathrm{Lie}_{f\Theta} h = \iota_{f\Theta}dh = f\cdot\iota_{\Theta}dh = f\cdot\mathrm{Lie}_{\Theta}h$

Prop^n/Def^n: The $\otimes$-product functor.

$$- \underset{\mathcal{O}_X}{\otimes} - : D^{\ell}(X) \otimes D^r(X) \longrightarrow D^r(X).$$

$$M \quad N \longmapsto M \otimes_{\mathcal{O}_X} N.$$

where M N:

$$(f \otimes \omega)\Theta = f \otimes (\omega\Theta) - (\Theta f) \otimes \omega.$$

has desired codomain

Pf: Exercise!

Corollary: Let $\omega_X = \Omega^n_X[n]$. (shift not important yet).

$$- \underset{\mathcal{O}_X}{\otimes} \omega_X : \underset{\underset{D^{\ell}(X)^{\heartsuit}}{\cup}}{D^{\ell}(X)} \xrightarrow{\simeq} \underset{\underset{D^r(X)^{\heartsuit}[n]}{\cup}}{D^r(X)}$$

exact up to shift $D^{\ell}(X)^{\heartsuit} \xrightarrow{\simeq} D^r(X)^{\heartsuit}[n]$

"left and right t structures agree (up to shift)".

$$\begin{tikzcd}
\mathbb{1} \arrow[r, mapsto] & \omega_X \arrow[r, mapsto, "\omega_X^{-1}"] & \mathcal{O}_X
\end{tikzcd}$$

$$\begin{tikzcd}
D(X) = D^r(X) \arrow[r, "\simeq"] \arrow[d, "o^r"] & D^{\ell}(X) \arrow[d, "o^{\ell}"] \\
QC(X) \arrow[r, "\omega_X^{-1}", "\simeq"'] & QC(X)
\end{tikzcd}$$

Philosophy:

$$\begin{tikzcd}
 & D(X) \arrow[dl, "o^r"'] \arrow[dr, "o^{\ell}"] & \\
QC(X) \arrow[rr, "\omega_X^{-1}"] & & QC(X)
\end{tikzcd}$$

To define functor, $D(X) \to D(Y)$, give $D^r(X) \to D^r(Y)$

or $D^r(X) \to D^{\ell}(X)$

or ... $\ell \to \ell$ or $\ell \to r$.

e.g.

$$\begin{tikzcd}
D^{\ell}(X)^{\otimes 2} \arrow[r, "- \otimes_{\mathcal{O}_X} -"] & D^{\ell}(X) \\
\overset{!}{\otimes} : D(X)^{\otimes 2} \arrow[r] \arrow[u] \arrow[d] & D(X) \arrow[u] \arrow[d] \\
D^r(X)^{\otimes 2} \arrow[r, "- \otimes_{\mathcal{O}_X} - \otimes_{\mathcal{O}_X} \omega_X^{-1}"] & D^r(X)
\end{tikzcd}$$

"left D-modules like functions pull back"

$\delta(\mathcal{F}) = (\delta\cdot\partial)(F)$..

"right D-module are distributions so they push forward"

Let $f : X \to Y$. let $d = d_x - d_y$ the relative dim^n.

$d > 0$ if $X \twoheadrightarrow Y$

$d < 0$ if $X \hookrightarrow Y$.

$$\begin{tikzcd}
f^! : D(Y) \arrow[r] \arrow[d, "\wr"] & D(X) \arrow[d, "\wr"] \\
f^{\bullet} \quad D^{\ell}(Y) \arrow[r] \arrow[d] & D^{\ell}(X) \arrow[d] \\
D^r(Y) \arrow[r] & D^r(X)
\end{tikzcd}$$

$$M \longmapsto f^{\bullet}(M \otimes_{\mathcal{O}_Y} \omega_Y^{\vee}) \otimes_{\mathcal{O}_X} \omega_X \sim f^{\bullet}(M)[d]$$

(Suppose ω_X, ω_Y trivialized.)

$$= f^{\bullet}M \otimes_{\mathcal{O}_X} \omega_{X/Y}$$

E.g. $f^!\underbrace{\omega_Y} = \omega_X$

in deg $-d_y$ in deg $-d_x$.

$F: X \to Y$ of rel. dim^{n} $d = d_X - d_Y$.

$$\begin{tikzcd} F_*: D(X) \arrow[r] \arrow[d, equal] & D(Y) \arrow[d, equal] \\ D^r(X) \arrow[r] & D^r(Y) \end{tikzcd}$$

Consider

$$\begin{tikzcd} D_Y \arrow[d, mapsto] & \in & D^l(Y) \arrow[d, "f^!"] \\ f^\bullet D_Y & \in & D^l(X) \end{tikzcd}$$

$$M \longmapsto f_\bullet\Big(\overbrace{M \underset{D_X}{\otimes} f^! D_Y}^{\overset{Sh(X)}{\cup}}\Big) \in D^r(Y). \quad \in (D_X, f^! D_Y)\text{-Mod} \text{ since}$$

$\underline{\text{Prop}^{n}}$: For $\iota: X \hookrightarrow Y$ a closed embedding (Also true for $j: X \to Y$ affine) $\iota_\bullet: D(X) \to D(Y)$ exact.

$\underline{\text{Pf}}$: $f^! D_Y$ l.f. over D_X, $f_\bullet$ is exact. ▨ $\left(\mathbb{C}[\partial_x] \simeq x^{-1}\mathbb{C}[x^{-1}]\right)$

Essentially, $M \longmapsto M \boxtimes \mathbb{C}[\partial_{x_i}]$ x_i normal coordinates.

'For D^l, " $M \longmapsto M \boxtimes \mathbb{C}[\partial_{x_i}]\,[\underset{\overset{\parallel}{-\text{codim} = d < 0 \text{ for closed emb.}}}{d_X - d_Y}]$

"shifted up by codim."

The Cousin complex:

$$\begin{tikzcd}
X_1 \arrow[r, hook, "\iota_1"] & X & X^0 \arrow[l, hook', "j^0"'] \\
X_2 \arrow[r, hook, "\tilde\iota_2"] \arrow[ur, "\iota_2"] & X_1 \arrow[u, hook, "\iota_1"] & X_1^0 \arrow[l, hook', "\tilde j_1"'] \arrow[ul, "j_1"'] \\
\arrow[u, "\iota_{k+1}"] & \arrow[u, hook] & \arrow[u, "j_k"'] \\
X_{k+1} \arrow[r, hook] & X_k & X_k^0 \arrow[l, hook'] \\
 & \vdots & \\
\{pts\} = X_n \arrow[r, hook] & X_{n-1} & X_{n-1}^0 \arrow[l, hook']
\end{tikzcd}$$

$M \hookrightarrow j_{0*}j_0^! M \twoheadrightarrow \iota_{1*}\iota_1^! M[1]$

$\iota_1^! M \hookrightarrow \tilde j_{1*}\tilde j_1^! \iota_1^! M \twoheadrightarrow \tilde\iota_{2*}\tilde\iota_2^! \iota_1^! M.$

$\iota_{1*}\iota_1^! M \hookrightarrow \iota_{1*}\tilde j_{1*}\tilde j_1^! \iota_1^! M \twoheadrightarrow \iota_{1*}\tilde\iota_{2*}\tilde\iota_2^! \iota_1^! M$

$\qquad\qquad\qquad\qquad \overset{\parallel}{j_{1*}j_1^! M} \twoheadrightarrow \overset{\parallel}{\iota_{2*}\iota_2^! M}$

$\rightsquigarrow \iota_{k*}\iota_k^! M \hookrightarrow j_{k*}j_k^! M \twoheadrightarrow \iota_{k+1*}\iota_{k+1}^! M.$

Enough for one board!

$\underline{\text{Prop}^{n}}$: Let $Z \underset{\text{closed}}{\overset{\iota}{\hookrightarrow}} X \underset{\text{open}}{\overset{j}{\hookleftarrow}} U = X \setminus Z$

Then $\iota_*\iota^! M \to M \to j_*j^! M$ is exact $\triangle$ in $D(X)$.

$\underline{\text{E.g.}}$: $\{\infty\} \hookrightarrow \mathbb{P}^1 \hookleftarrow \mathbb{A}^1$, $M = \mathcal{O}_{\mathbb{P}^1} \in D^l(\mathbb{P}^1)$.

$$\iota_*\mathcal{O}_{\{\infty\}} \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow j_*\mathcal{O}_{\mathbb{A}^1} \qquad w^{-1}\mathbb{C}[w^{-1}]$$

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}[\overset{\overset{w^{-1}}{\parallel}}{z}] \longrightarrow \overset{\parallel}{z\mathbb{C}[z]} \longrightarrow 0.$$

$$\begin{tikzcd}
 & h & & & \\
e = w^2\partial_w & 0 & \mathbb{C} \arrow[r] & \mathbb{C}_{z^0} & \\
f = -\partial_w & -2 & & \mathbb{C}_{z^1} \arrow[u, bend right] \arrow[r] \arrow[d, bend right] & \mathbb{C}_{z^1} \arrow[d, bend right] \\
h = 2w\partial_w & -4 & & \mathbb{C}_{z^2} \arrow[r] \arrow[u, bend right] & \mathbb{C}_{z^2} \arrow[u, bend right]
\end{tikzcd}$$

"Taking the singularity type".

---

0 1 2 3.

$$\begin{tikzcd}
M \arrow[d, hook] & & & \\
j_{0*}j_0^! M \arrow[r, two heads] \arrow[dr, "d_0"'] & \overset{\overset{\mathrm{im}(d_0)}{\parallel}}{\iota_{1*}\iota_1^! M} = \ker(d_1) \arrow[d, hook] & & \\
 & j_{1*}j_1^! M \arrow[r, two heads] \arrow[dr, "d_1"'] & \overset{\overset{\mathrm{im}(d_1).}{\parallel}}{\iota_{2*}\iota_2^! M} = \ker(d_2) \arrow[d, hook] & \vdots \\
 & & j_{2*}j_2^! M \arrow[r, two heads] & \iota_{3*}\iota_3^! M \arrow[d, equal, "d_2."']
\\
 & & & j_{3*}j_3^! M
\end{tikzcd}$$

$$\begin{tikzcd}
 & M \arrow[d, hook] & & & & \\
\Longrightarrow & j_{0*}j_0^! M \arrow[r, "d_0"] & j_{1*}j_1^! M \arrow[r, "d_1"] & j_{2*}j_2^! M \arrow[r, "d_2."] & j_{3*}j_3^! M \arrow[r] & 0
\end{tikzcd}$$

is exact.

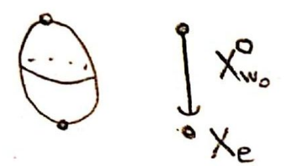
$e, x, z, v, y, w_\circ = e, s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_1}s_{\alpha_2}, s_{\alpha_2}s_{\alpha_1}?, s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}$

Recall: $X_w = \overline{X_w}$

$$X = \bigsqcup_{w \in W} X_w^\circ, \quad X_w^\circ = Bw_\circ B/_B \simeq \mathbb{C}^{l(w)}$$

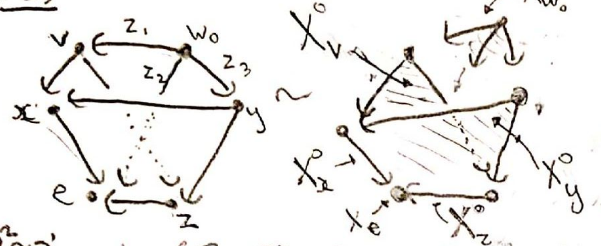
$w \leq y$ iff $X_w^\circ \subset X_y$, $X = X_{w_\circ}$

SL_2



$\mathbb{P}^1 = pt \sqcup \mathbb{C}$

SL_3



$\mathbb{P}^2 \tilde{\times} \mathbb{P}^1 = pt \sqcup (\mathbb{C} \sqcup \mathbb{C}) \sqcup (\mathbb{C}^2 \sqcup \mathbb{C}^2) \sqcup \mathbb{C}^3$

$$\begin{tikzcd} X \arrow[r, phantom, "\overset{\shortparallel}{X_1}"] & X^\circ_{w_\circ} \arrow[r, hook] & X & X^\circ_{w_\circ} \arrow[l, hook'] \arrow[r, phantom, "= \mathbb{C}^3"] & {} \\ & X_2 \arrow[r, hook] & X_1 \arrow[u, hook] & X^\circ_v \sqcup X^\circ_y \simeq \mathbb{C}^2 \sqcup \mathbb{C}^2 \arrow[l, hook'] & v, y \in W^{(2)} \\ & pt. = X_3 \arrow[r, hook] & X_2 \arrow[u, hook] & X^\circ_z \sqcup X^\circ_x = \mathbb{C} \sqcup \mathbb{C} \arrow[l, hook'] & z, x \in W^{(1)} \end{tikzcd}$$

$$\mathcal{N}_w = j_{w*} j_w^! \mathcal{O}_X$$

$$= j_{w*} \mathcal{O}_{X^\circ_w}$$

$$\begin{tikzcd}[column sep=small] L(-w_\circ\rho-\rho) \arrow[d, phantom, "\shortparallel"] & M(\overset{0}{\overset{\shortparallel}{-w_\circ\rho}}-\rho)^\vee \arrow[r] & M(-\overset{\tilde{y}\cdot 0}{y\rho}-\rho)^\vee \arrow[d, phantom, "\wr"] & M(-\overset{\tilde{z}\cdot 0}{z\rho}-\rho)^\vee \arrow[d, phantom, "\shortparallel"] & \overset{w_\circ \cdot 0}{\overset{\shortparallel}{M(-2\rho)^\vee}} \arrow[d, phantom, "\shortparallel"] \\ \mathcal{O}_X \arrow[r, hook] & j_{w_\circ *}\mathcal{O}_{X^\circ_{w_\circ}} \arrow[r] & j_{y*}\mathcal{O}_{X^\circ_y} \oplus j_{v*}\mathcal{O}_{X^\circ_v} \arrow[r] & j_{z*}\mathcal{O}_{X^\circ_z} \oplus j_{x*}\mathcal{O}_{X^\circ_x} \arrow[r] & i_{[e]*}\mathbb{C} \end{tikzcd}$$

$l(\tilde{y}) = 1$ $\quad$ $l(\tilde{z}) = 2$

" $\delta_0 \boxtimes \mathcal{O}_{\mathbb{C}^2}$ " $\quad$ $\delta_{(0,0)} \boxtimes \mathcal{O}_{\mathbb{C}}$

$\Gamma(X, -)$.

$$\begin{tikzcd}[column sep=small] \mathbb{C} \arrow[d] & \left(w_1^{-1}\mathbb{C}[v_1^{-1}] \otimes_{\mathbb{C}} \mathbb{C}[z_2, z_3]\right) \arrow[d, phantom, "\shortparallel"] & & \\ \mathbb{C}[z_1, z_2, z_3] \arrow[r, Rightarrow] & z_1\mathbb{C}[z_1,z_2,z_3] \oplus z_3\mathbb{C}[z_1,z_2,z_3] \arrow[r] & z_1z_2\mathbb{C}[z_1,z_2,z_3] \cong z_3z_1\mathbb{C}[z_1,z_2,z_3] \oplus z_3z_1\mathbb{C}[z_1,z_3,z_3] \cong z_2z_3\mathbb{C}[z_1,z_2,z_3] \arrow[r] & z_1z_2z_3\mathbb{C}[z_1,z_2,z_3] \end{tikzcd}$$

Let $\lambda \in -P_+^{reg}$, $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{<0}$ $\forall \alpha^\vee \in \Phi_+$.

Let $D_\lambda = \mathrm{Diff}(\mathcal{L}(\lambda+\rho))$

Th^m [BB., B.K.] The functor $\Gamma : D_\lambda(X) \xrightarrow{\Gamma} \Gamma(X, D_\lambda)\text{-Mod}$ is an exact equivalence.

$$\begin{tikzcd} D_\lambda(X) \arrow[r, "\Gamma"] & \Gamma(X, D_\lambda)\text{-Mod} \arrow[d, "\wr"] \\ & \mathfrak{g}\text{-Mod}_\lambda \arrow[ul, "\mathbb{L}"] \end{tikzcd}$$

We have

$$U\mathfrak{g} \cdot \mathrm{Ker}(\chi_\lambda) \hookrightarrow U\mathfrak{g} \xrightarrow{\rho} \Gamma(X, D_\lambda).$$

So that

$$\Gamma(X, D_\lambda)\text{-Mod} \xrightarrow{\simeq} \mathfrak{g}\text{-Mod}_\lambda.$$

and the inverse is

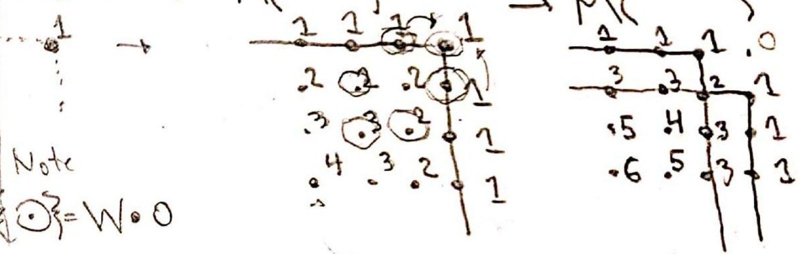
$$\mathbb{L} = D_\lambda \otimes_{U\mathfrak{g}} (-) : \mathfrak{g}\text{-Mod}_\lambda \longrightarrow D_\lambda(X).$$

Note " " $\mathcal{O}_X \longmapsto M(-w\rho-\rho)^\vee$

$$\mathcal{N}_w = j_{w*}j_w^! \mathcal{L}(\lambda+\rho) \longmapsto M(w\cdot(\lambda-\rho))^\vee$$

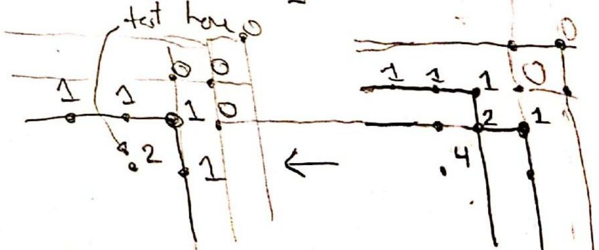
In this formalism. $M(\lambda) \in \mathfrak{g}\text{-Mod}_{\lambda-\rho}$ $\quad$ $M(0) \in \mathfrak{g}\text{-Mod}_{-\rho}$

$L(0) \longrightarrow M(\quad)^\vee \quad \dfrac{\oplus M(\quad)^\vee}{} \longrightarrow M(\quad)^\vee$



Note $\{\odot\} = W \cdot 0$

$$[4 - 6 + 4 - 2 = 0] \checkmark.$$



$$h_{\alpha_1} = -2z_1\partial_{z_1} + z_2\partial_{z_2} + z_3\partial_{z_3}$$

$$h_{\alpha_2} = z_1\partial_{z_1} - 2z_2\partial_{z_2} - 2z_3\partial_{z_3}.$$

$e_{\alpha_1} = -\partial_{z_1}$ $\quad$ $+(z_2 - z_1z_3)\partial_3$ ✓

$f_{\alpha_1} = z_1^2\partial_{z_1} - z_1z_2\partial_{z_2}$

$e_{\alpha_2} = z_1\partial_{z_2} - \partial_{z_3}$

$f_{\alpha_2} = z_2\partial_{z_1} + z_3^2\partial_{z_3}$