1 $\mathcal{D}$-module formalism

Let X be a smooth variety of dimension d_X over $\mathbb{K} = \mathbb{C}$ or a field of characteristic 0. We write $\mathcal{O}_X$ for the sheaf of regular functions, $\mathcal{D}_X$ for the sheaf of differential operators, Θ_X for the tangent sheaf, Ω^1_X for the sheaf of Kahler differentials, $\Omega^{d_X}_X$ for the sheaf of sections of the canonical bundle, and $\omega_X = \Omega^{d_X}_X[d_X]$ for the dualizing sheaf on X.

Let $\mathrm{Sh}(X)$ denote the DG category of complexes of sheaves of $\mathbb{K}$-modules on X.

1.1 $\mathcal{O}$-module conventions

Let $\mathcal{O}_X$-Mod be the DG category of complexes of $\mathcal{O}_X$-modules, and $\mathrm{QC}(X)$ and $\mathrm{Coh}(X)$ be the full sub DG categories of complexes of $\mathcal{O}_X$-modules with quasi-coherent and coherent cohomology.

The category $\mathcal{O}_X$-Mod is symmetric monoidal with respect to the tensor product $\otimes_{\mathcal{O}_X}$, with unit object $\mathcal{O}_X$, and $\mathrm{QC}(X)$ and $\mathrm{Coh}(X)$ are monoidal subcategories.

For $f : X \to Y$ a map of schemes, we define the inverse and direct image functors by

$$f^\bullet : \mathcal{O}_Y\text{-Mod} \to \mathcal{O}_X\text{-Mod} \qquad f^\bullet\mathcal{F} = f^{-1}F \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \qquad \text{and} \qquad f_\bullet : \mathcal{O}_X\text{-Mod} \to \mathcal{O}_Y\text{-Mod} \qquad f_\bullet\mathcal{F} = f_\bullet\mathcal{F} \ ,$$

where $f_\bullet : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ and $f^{-1} : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ are the usual direct and inverse image functors on sheaves of $\mathbb{K}$-modules; note that $f^\bullet$ preserves quasicoherence, as does $f_\bullet$ for quasicompact, quasiseperated maps. We define the global sections functor by $\Gamma = \pi_\bullet : \mathcal{O}_X\text{-Mod} \to \mathrm{Vect}$ where $\pi : X \to \mathrm{pt}$.

For $\mathcal{F}, \mathcal{G} \in \mathcal{O}_X$-Mod, we define the internal hom object

$$\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \in \mathcal{O}_X\text{-mod} \qquad \text{by} \qquad \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) := \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U) \ ,$$

noting that for $\mathcal{H} \in \mathcal{O}_X$-Mod, we have

$$\mathrm{Hom}(\mathcal{H}, \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})) \cong \mathrm{Hom}(\mathcal{H} \otimes_{\mathcal{O}_X} \mathcal{F}, \mathcal{G}) \ .$$

In particular, the space of homomorphisms is given by the space of sections of the internal hom object

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}(\mathcal{O}_X, \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})) = \Gamma(X, \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})).$$

If $\mathcal{F} \in \mathrm{Coh}(X)$ is coherent and $\mathcal{G} \in \mathrm{QC}(X)$ is quasi-coherent, then $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \in \mathrm{QC}(X)$ is quasi-coherent. If $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}(X)$ are both coherent, then $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \in \mathrm{Coh}(X)$ is also coherent

The duality functor on coherent $\mathcal{O}_X$-modules is defined by

$$(-)^\vee : \mathrm{Coh}(X) \to \mathrm{Coh}(X) \qquad \text{by} \qquad \mathcal{F} \mapsto \mathcal{F}^\vee := \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X).$$

One has canonical isomorphisms $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \cong \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F}^\vee$ and $(\mathcal{F}^\vee)^\vee \cong \mathcal{F}$.

1.2 $\mathcal{D}$-module conventions

Let $\mathrm{D}^l(X)$ and $\mathrm{D}^r(X)$ be the concrete DG categories of complexes of left and right $\mathcal{D}_X$-modules which are quasicoherent as $\mathcal{O}_X$-modules, and let $\mathrm{D}^l_{\mathrm{coh}}(X)$ and $\mathrm{D}^r_{\mathrm{coh}}(X)$ denote the full sub DG categories of complexes with cohomology that is coherent as a module over $\mathcal{D}_X$.

Example 1.1. The sheaf of regular functions $\mathcal{O}_X \in \mathrm{D}^l(X)^\heartsuit$ has the structure of a left $\mathcal{D}_X$ module, given by the defining action of the sheaf of differential operators $\mathcal{D}_X$ on $\mathcal{O}_X$.

More generally, a left $\mathcal{D}_X$ module (or a complex of such) $M \in \mathrm{D}^l(X)$ is given by a quasicoherent sheaf (or a complex of such) $M \in \mathrm{QC}(X)$, together with a flat connection, that is, $\nabla \in \mathrm{Hom}_{\mathrm{Sh}(X)}(M, \Omega^1_X \otimes_{\mathcal{O}_X} M)$ such that

- $\nabla_\theta(fs) = \theta(f)s + f\nabla_\theta(s)$, and
- $\nabla_{[\theta_1,\theta_2]}s = [\nabla_{\theta_1}, \nabla_{\theta_2}]s$,

where $\theta, \theta_1, \theta_2 \in \Theta_X$, $f \in \mathcal{O}_X$, and $s \in M$. The first condition is that ∇ defines a connection, and the second that ∇ is flat.

Example 1.2. The sheaf of sections of the canonical bundle $\Omega^{d_X}_X \in \mathrm{D}^r(X)^\heartsuit$ is the protypical example of a right $\mathcal{D}_X$ module, with action of vector fields given by $\theta(\eta) = -\mathrm{Lie}_\theta(\eta)$ for $\theta \in \Theta_X$ and $\eta \in \Omega^{d_X}_X$.

There is a canonical equivalence of the categories $\mathrm{D}^l(X)$ and $\mathrm{D}^r(X)$

$$\mathrm{D}^l(X) \underset{(-)^l}{\overset{(-)^r}{\rightleftarrows}} \mathrm{D}^r(X) \qquad \text{defined by} \qquad \begin{cases} M \mapsto M^l := M \otimes_{\mathcal{O}_X} \omega_X^\vee & \text{for } M \in \mathrm{D}^r(X) \quad \text{and} \\ L \mapsto L^r := \omega_X \otimes_{\mathcal{O}_X} L & \text{for } L \in \mathrm{D}^l(X). \end{cases}$$

We write $\mathrm{D}(X)$ for the abstract DG category given by the common value of $\mathrm{D}^r(X)$ and $\mathrm{D}^l(X)$ under this identification, and $\mathrm{D}_{\mathrm{coh}}(X)$ for the full sub DG category corresponding to $\mathrm{D}^r_{\mathrm{coh}}(X)$ and $\mathrm{D}^l_{\mathrm{coh}}(X)$, which are also identified under this equivalence. $\mathrm{D}^r(X)$ and $\mathrm{D}^l(X)$ both have natural forgetful functors to $\mathrm{QC}(X)$, which are intertwined by tensoring with ω_X. This perspective is summarized in the following diagram:

$$\begin{array}{ccc} \mathrm{D}^l(X) & \xrightarrow[\simeq]{\omega_X} & \mathrm{D}^r(X) \\ \downarrow{\scriptstyle o^l} & & \downarrow{\scriptstyle o^r} \\ \mathrm{QC}^l(X) & \xrightarrow[\simeq]{\omega_X} & \mathrm{QC}^r(X) \end{array} \qquad \text{so that} \qquad \begin{array}{ccc} \mathrm{D}(X) & & \\ \downarrow{\scriptstyle o^l} & \searrow^{o^r} & \\ \mathrm{QC}(X) & \xrightarrow[\simeq]{\omega_X} & \mathrm{QC}(X) \end{array}$$

Throughout, when defining a functor involving (potentially several copies of) the category $\mathrm{D}(X)$, we will prescribe the values of the functor in terms of a particular choice of realization $\mathrm{D}^r(X)$ or $\mathrm{D}^l(X)$ for each copy of $\mathrm{D}(X)$, with the extension to all other choices of concrete realizations of $\mathrm{D}(X)$ implicitly specified via the above equivalence.

Note that the above equivalence is exact up to a cohomological degree shift of $d_X = \dim_{\mathbb{K}} X$, so that the category $\mathrm{D}(X)$ inherits two different t-structures, which differ only by this shift. We choose to preference the right t structure, and all statements about exactness of functors involving $\mathrm{D}(X)$ will be given in these terms. After fixing our conventions for the six functors formalism below, this t structure will be the one which corresponds to the perverse t structure on constructible sheaves under the Riemann-Hilbert correspondence. In particular, under this identification $\omega_X \in \mathrm{D}(X)$ is the dualizing sheaf, $\omega_X[-d_X] \in \mathrm{D}(X)^\heartsuit$ is the IC sheaf, and $\omega_X[-2d_X] \in \mathrm{D}(X)$ is the constant sheaf.

1.3 The six functors formalism for $\mathcal{D}$-modules

The category $\mathrm{D}(X)$ is symmetric monoidal with respect to $\otimes^! : \mathrm{D}(X)^{\otimes 2} \to \mathrm{D}(X)$ defined by

$$\otimes^! : \mathrm{D}^l(X) \otimes \mathrm{D}^l(X) \to \mathrm{D}^l(X) \qquad M \otimes^! N = M \otimes_{\mathcal{O}_X} N \quad \text{with} \quad P(m \otimes n) = Pm \otimes n + m \otimes Pn$$

for $P \in \mathcal{D}_X$; this formula agrees with the usual definition of the tensor product of connections, and tensor products of flat connections are flat. The corresponding functor $\otimes^! : \mathrm{D}^r(X) \otimes \mathrm{D}^r(X) \to \mathrm{D}^r(X)$ is given by $M \otimes^! N = M \otimes_{\mathcal{O}_X} N \otimes_{\mathcal{O}_X} \omega_X^\vee$. We let $\mathbb{1} \in \mathrm{D}(X)$ denote the tensor unit, and note $o^l(\mathbb{1}) = \mathcal{O}_X$ and $o^r(\mathbb{1}) = \omega_X$. We will often use just $\otimes$ to denote this symmetric monoidal structure.

Let $f : X \to Y$ be a map of smooth varieties. We define the inverse image functor $f^! : \mathrm{D}(Y) \to \mathrm{D}(X)$ by

$$f^! : \mathrm{D}^l(Y) \to D^l(X) \qquad f^!(M) = f^\bullet(M) \quad \text{equipped with the pullback flat connection.}$$

This functor is symmetric monoidal with respect to $\otimes^!$, and in particular maps the tensor unit $\mathbb{1}_Y$ to $\mathbb{1}_X$. The corresponding functor $f^! : \mathrm{D}^r(X) \to \mathrm{D}^r(X)$ is given by $f^!(M) = f^\bullet(M \otimes_{\mathcal{O}_Y} \omega_Y^\vee) \otimes_{\mathcal{O}_X} \omega_X$.

We define the exterior product

$$\boxtimes : \mathrm{D}(X) \otimes \mathrm{D}(Y) \to \mathrm{D}(X \times Y) \qquad \text{by} \qquad M \boxtimes N = \pi_X^! M \otimes \pi_Y^! N \ ,$$

for $\pi_X : X \times Y \to X, \pi_Y : X \times Y \to Y$. Note that

$$M \otimes N = \Delta^!(M \boxtimes N)$$

for $M, N \in \mathrm{D}(X)$ and $\Delta : X \to X \times X$ the diagonal embedding.

Let $f : X \to Y$ again be a map of smooth varieties. We define the direct image functor $f_* : \mathrm{D}(X) \to \mathrm{D}(Y)$ by

$$f_* : \mathrm{D}^r(X) \to D^r(Y) \qquad f_*(M) = f_\bullet(M \otimes_{\mathcal{D}_X} \mathcal{D}_{X \to Y}) \qquad \text{for} \qquad \mathcal{D}_{X \to Y} := f^! \mathcal{D}_Y \in (\mathcal{D}_X, f^{-1}\mathcal{D}_Y)\text{-Mod}$$

where $\mathcal{D}_{X \to Y} = f^! \mathcal{D}_Y \in \mathrm{D}^l(X)$ is defined in terms of $\mathcal{D}_Y \in \mathrm{D}^l(Y)$ as a left module, so that the additional $(\mathcal{D}_Y, \mathcal{D}_Y)$-bimodule structure on $\mathcal{D}_Y$ equips $\mathcal{D}_{X \to Y}$ with the structure of a $(\mathcal{D}_X, f^{-1}\mathcal{D}_Y)$-bimodule.

We define the de Rham cohomology functor $\Gamma_{\mathrm{dR}} := \pi_* : \mathrm{D}(X) \to \mathrm{Vect}$, where $\pi : X \to \mathrm{pt}$. Note that we have

$$\Gamma_{\mathrm{dR}} : \mathrm{D}^r(X) \to \mathrm{Vect} \quad \Gamma_{\mathrm{dR}}(M) = \pi_\bullet(M \otimes_{\mathcal{D}_X} \mathcal{O}_X) \qquad \text{and} \qquad \Gamma_{\mathrm{dR}} : \mathrm{D}^l(X) \to \mathrm{Vect} \quad \Gamma_{\mathrm{dR}}(M) = \pi_\bullet(\omega_X \otimes_{\mathcal{D}_X} M) \; .$$

For $M, N \in \mathrm{D}(X)$, we define the sheaf internal hom functor

$$\underline{\mathrm{Hom}}_{\mathrm{D}(X)}(\cdot, \cdot) : \mathrm{D}(X)^{\mathrm{op}} \otimes \mathrm{D}(X) \to \mathrm{Sh}(X) \qquad \text{by} \qquad \underline{\mathrm{Hom}}_{\mathrm{D}(X)}(M, N)(U) = \mathrm{Hom}_{\mathrm{D}(U)}(j^! M, j^! N) \; ,$$

where $j : U \to X$ is the open embedding, and note that

$$\Gamma \circ \underline{\mathrm{Hom}} = \mathrm{Hom} : \mathrm{D}(X)^{\mathrm{op}} \otimes \mathrm{D}(X) \to \mathrm{Vect} \qquad \Gamma(X, \underline{\mathrm{Hom}}_{D(X)}(M, N)) = \mathrm{Hom}_{D(X)}(M, N) \; .$$

We define the duality functor $\mathbb{D} : \mathrm{D}_{\mathrm{coh}}(X)^{\mathrm{op}} \to \mathrm{D}(X)$ by

$$\mathbb{D} : \mathrm{D}^r_{\mathrm{coh}}(X)^{\mathrm{op}} \to \mathrm{D}^l_{\mathrm{coh}}(X) \qquad \mathbb{D}(M) = \underline{\mathrm{Hom}}_{\mathrm{D}^r(X)}(M, \mathcal{D}_X) \; ,$$

where $\mathcal{D}_X \in \mathrm{D}^r(X)$ is considered as a $(\mathcal{D}_X, \mathcal{D}_X)$-bimodule so that $\mathbb{D}(M)$, which is a priori an object in $\mathrm{Sh}(X)$, defines an object of $\mathrm{D}^l(X)$ as desired. Note that $\mathbb{D}$ preserves coherence, but if M is not coherent, then the resulting object of $\mathcal{D}_X$-Mod is not in general quasicoherent as an object of $\mathcal{O}_X$-Mod.

We define the genuine internal hom functor $\mathcal{H}\mathrm{om}(\cdot, \cdot) : \mathrm{D}_{\mathrm{coh}}(X)^{\mathrm{op}} \otimes \mathrm{D}(X) \to \mathrm{D}(X)$ by

$$\mathcal{H}\mathrm{om}(\cdot, \cdot) : \mathrm{D}^r_{\mathrm{coh}}(X)^{\mathrm{op}} \otimes \mathrm{D}^l(X) \to \mathrm{D}^l(X) \qquad \mathcal{H}\mathrm{om}(M, N) = \underline{\mathrm{Hom}}_{\mathrm{D}^r(X)}(M, N \otimes_{\mathcal{O}_X} \mathcal{D}_X) \; ,$$

where $N \otimes_{\mathcal{O}_X} \mathcal{D}_X \in \mathrm{D}^r(X)$ is considered as a $(\mathcal{D}_X, \mathcal{D}_X)$-bimodule so that $\mathcal{H}\mathrm{om}(M, N) \in \mathrm{D}^l(X)$ as above. Note that

$$\Gamma_{\mathrm{dR}} \circ \mathcal{H}\mathrm{om} = \mathrm{Hom} : \mathrm{D}_{\mathrm{coh}}(X)^{\mathrm{op}} \otimes \mathrm{D}(X) \to \mathrm{Vect} \qquad \Gamma_{\mathrm{dR}}(X, \mathcal{H}\mathrm{om}_{D(X)}(M, N)) = \mathrm{Hom}_{D(X)}(M, N) \; .$$

Further, we have

$$\mathcal{H}\mathrm{om}(\cdot, \cdot) = \mathbb{D}(\cdot) \otimes^! (\cdot) : \mathrm{D}_{\mathrm{coh}}(X)^{\mathrm{op}} \otimes \mathrm{D}(X) \to \mathrm{D}(X) \; ,$$

and in particular $\mathcal{H}\mathrm{om}(\cdot, \mathbb{1}) = \mathbb{D} : \mathrm{D}(X)^{\mathrm{op}} \to \mathrm{D}(X)$; we could equivalently take this as the definition of $\mathcal{H}\mathrm{om}$.

The pushforward and pullback functors f_* and $f^!$ above were defined on the entire category $\mathrm{D}(X)$, but their putative adjoints can not always be defined. In general, the best we can do is the following: Let $f : X \to Y$ again be a map of smooth varieties, and let $\mathrm{D}^{f^!}_{\mathrm{coh}}(Y)$ be the full subcategory of objects $M \in \mathrm{D}_{\mathrm{coh}}(Y)$ such that $f^! \mathbb{D} M \in \mathrm{D}_{\mathrm{coh}}(X)$ is coherent, and similarly $\mathrm{D}^{f_*}_{\mathrm{coh}}(X)$ be the full subcategory of objects $M \in \mathrm{D}_{\mathrm{coh}}(X)$ such that $f_* \mathbb{D} M \in \mathrm{D}_{\mathrm{coh}}(Y)$ is coherent. Then we define

$$f^* := \mathbb{D} f^! \mathbb{D} : \mathrm{D}^{f^!}_{\mathrm{coh}}(Y) \to \mathrm{D}_{\mathrm{coh}}(X) \qquad f_! := \mathbb{D} f_* \mathbb{D} : \mathrm{D}^{f_*}_{\mathrm{coh}}(X) \to \mathrm{D}_{\mathrm{coh}}(Y) \; .$$

In various situations, these definitions simplify to more useful ones:

Suppose $f : X \to Y$ is smooth map of relative dimension $d = d_X - d_Y$ of smooth varieties. Then $f^! : \mathrm{D}_{\mathrm{coh}}(Y) \to \mathrm{D}_{\mathrm{coh}}(X)$ preserves coherence, so that $f^* : \mathrm{D}_{\mathrm{coh}}(Y) \to \mathrm{D}_{\mathrm{coh}}(X)$ is defined. Moreover, in this case $f^* = f^![-2d]$, and we have a natural isomorphism

$$\mathrm{Hom}_{\mathrm{D}(X)}(f^* M, N) = \mathrm{Hom}_{\mathrm{D}(Y)}(M, f_* N)$$

of functors $\mathrm{D}_{\mathrm{coh}}(X) \otimes \mathrm{D}(Y) \to \mathrm{Vect}$.

Suppose $f : X \to Y$ is proper map of smooth varieties. Then $f_* : \mathrm{D}_{\mathrm{coh}}(X) \to \mathrm{D}_{\mathrm{coh}}(Y)$ preserves coherence, so that $f_! : \mathrm{D}_{\mathrm{coh}}(X) \to \mathrm{D}_{\mathrm{coh}}(Y)$ is defined. Moreover, in this case $f_! = f_*$ and we have a natural isomorphism

$$\mathrm{Hom}_{\mathrm{D}(Y)}(f_! M, N) = \mathrm{Hom}_{\mathrm{D}(X)}(M, f^! N)$$

of functors $\mathrm{D}_{\mathrm{coh}}(X) \otimes \mathrm{D}(Y) \to \mathrm{Vect}$.