①

$$\simeq \Gamma(\mathbb{C}^2, T_{\mathbb{C}^2})^{\mathbb{C}^\times}$$
$$\cap$$
$$GL_2\mathbb{C} \curvearrowright \mathbb{C}^2_{x,y} \rightsquigarrow \mathfrak{gl}_2\mathbb{C} \to \Gamma(\mathbb{C}^2, T_{\mathbb{C}^2}).$$

$$\begin{bmatrix}1&0\\0&0\end{bmatrix} \mapsto x\partial_x, \quad \begin{bmatrix}0&0\\0&1\end{bmatrix} \mapsto y\partial_y, \quad \begin{bmatrix}0&1\\0&0\end{bmatrix} \mapsto x\partial_y, \quad \begin{bmatrix}0&0\\1&0\end{bmatrix} \mapsto y\partial_x.$$

$$\Gamma(\mathbb{C}^2, T_{\mathbb{C}^2}) \curvearrowright \Gamma(\mathbb{C}^2, \mathcal{O}_{\mathbb{C}^2}) = \bigoplus_{k\geq 0} \underbrace{\mathrm{Sym}^k(\mathbb{C}^2_{x,y})}_{V^k}$$

$$\mathfrak{sl}_2\mathbb{C}$$
$$\cap$$
$$\mathfrak{gl}_2\mathbb{C} \curvearrowright V^k = \{x^k, x^{k-1}y, x^{k-2}y^2, \dots, x^1y^{k-1}, y^k\}.$$

$x\partial_y = e$, $y\partial_x = f$

$$x\partial_x + y\partial_y \mapsto k\cdot \mathbb{1}_{V^k}$$
$$h = x\partial_x - y\partial_y \mapsto k,\ k-2,\ k-4,\ \dots,\ -k$$

Prop^{n}: $V^k = L(k)$ the $(k+1)$-dim^{l} irrep.

② Geometric Interpretation

$$L(k) = \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k)) \quad \text{[Borel-Weil-Bott]}$$
$$\cup$$
$$\mathfrak{sl}_2\mathbb{C} = \Gamma(\mathbb{C}^2, T_{\mathbb{C}^2})^{\mathbb{C}^\times} / (x\partial_x + y\partial_y) \simeq \Gamma(\mathbb{P}^1, T_{\mathbb{P}^1}).$$

① Can we generalize this to other rep^{n}s in $\mathcal{O}$? Vermas.

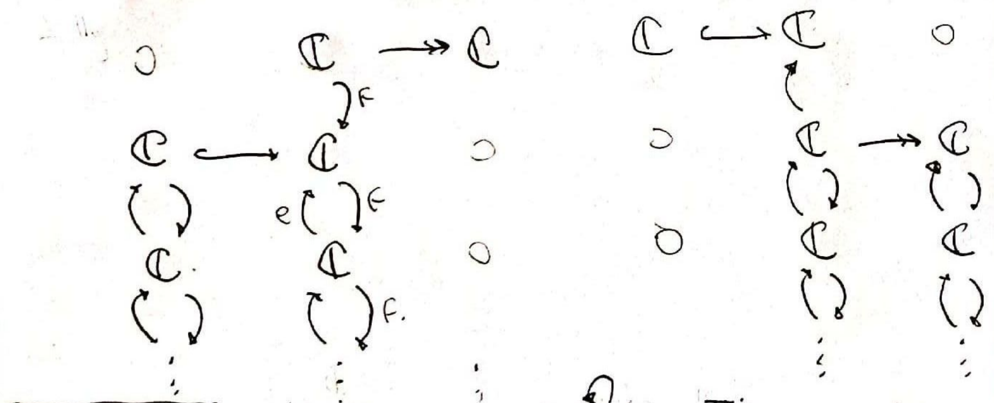
② Can we understand the structure of cat $\mathcal{O}$ geometrically?

③ What is the story for more general groups / Lie algebras?

① D-modules ② Six-functors — Riemann-Hilbert. ③ Flag varieties, B.B.

③ Preview of the answer for SL_2: $L(0) \cong L(0)^\vee$ "co-Vermas"

$$M(-2) \hookrightarrow M(0) \twoheadrightarrow L(0) \qquad L(0)^\vee \hookrightarrow M(0)^\vee \twoheadrightarrow M(-2)^\vee$$



$$\mathbb{C} \hookrightarrow \mathbb{C}[z] \overset{\pi}{\twoheadrightarrow} z\mathbb{C}[z]$$
$$\Gamma(\mathcal{O}_{\mathbb{P}^1}) \hookrightarrow \Gamma(j_*\mathcal{O}_{\mathbb{C}}) \twoheadrightarrow \Gamma(\delta_\infty)$$

global fn's; global fn's sing. at ∞; singularity types at ∞

$z = x/y$, $y=-1$
$e = -\partial_z$
$f = z^2\partial_z$
$h = -2z\partial_z$

④ Let X be a (smooth) algebraic variety.

$\mathcal{O}_X$ the sheaf of regular fn's. (sheaf of comm. alg's)

T_X the tgt sheaf (sheaf of Lie alg's)

D_X the sheaf of differential operators on X.

$$D_X \subset \underline{\mathrm{End}}_{\mathbb{C}_X}(\mathcal{O}_X)$$
$$= \left\langle \begin{matrix} f \in \mathcal{O}_X \simeq \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X), \\ \theta \in T_X \hookrightarrow \underline{\mathrm{End}}_{\mathbb{C}_X}(\mathcal{O}_X) \end{matrix} \;\middle|\; \begin{matrix} [\theta_x, \theta_y] = \theta_{[x,y]} \\ \theta_x(f\cdot g) = \theta_x(f)g + f\cdot\theta_x(g) \end{matrix} \right\rangle$$

"$D_X \sim$ universal enveloping algebra of T_X over $\mathcal{O}_X$."

$D(X)$ = (derived) category of (complexes of) sheaves of (left) modules over D_X.

①

Example: X ... z T_X

Then $D_X \circlearrowright \mathcal{O}_X$ by def. $\mathcal{O}_X \in D(X)$.

$D_X \circlearrowright D_X$ by left-mult. $D_X \in D(X)$.

$D_X \circlearrowright D_X \otimes_{\mathcal{O}_X} N =: {}_D N \leftarrow$ "Induced D-Mods" ${}_D(\ ): QC(X) \rightleftarrows D(X)$: for

$$\mathrm{Hom}_{D(X)}({}_D N, {}_D M) = \mathrm{Hom}_{QC(X)}(N, {}_D M)$$
$$= \Gamma(X, \underbrace{N^\vee \otimes_{\mathcal{O}_X} D_X \otimes_{\mathcal{O}_X} M}) \quad QC(X) \ni \mathrm{Diff}(N,M)$$
$$= \mathrm{Diff}(N,M)$$

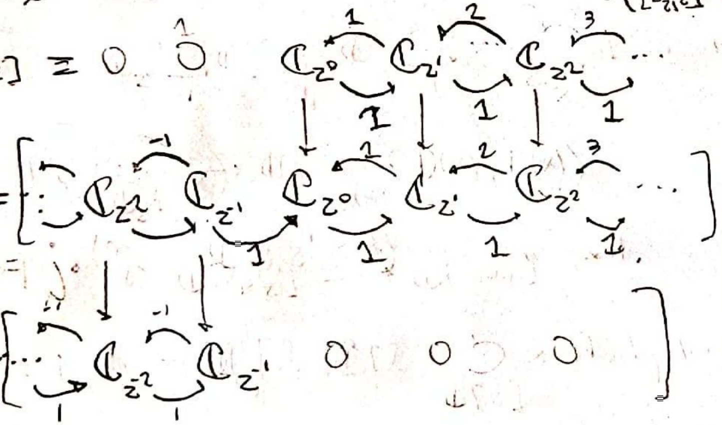
$E_P = [D \xrightarrow{P} D] \in D(X)$, $\rightsquigarrow \ker(P), \mathrm{coker}(P), \in D(X)$.

② $X = \mathbb{C}$, $\mathcal{O}_X = \mathbb{C}[z]$, $T_X = \mathbb{C}[z]\cdot\partial_z$, $D_X = \mathbb{C}\langle z, \partial_z\rangle / ([\partial_z, z] - 1)$

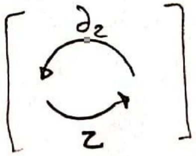
$\mathcal{O}_X = \mathbb{C}[z] \cong$

$j_* \mathcal{O}_{\mathbb{C}^\times} = \mathbb{C}[z, z^{-1}] =$

$\delta_0 = z^{-1}\mathbb{C}[z^{-1}] =$



Note: we found short exact sequence.



$$\mathcal{O}_{\mathbb{C}} \hookrightarrow j_* \mathcal{O}_{\mathbb{C}^\times} \longrightarrow \delta_0$$

regular fcs. — fcs w. poles at 0. — singularity types at 0. (= equivalence classes!)

③ Let $f: X \to Y$

$f^!: D(Y) \longrightarrow D(X)$

$M \longmapsto f^\circ(M) := \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}M \quad (D_X \circlearrowright \mathcal{O}_X)$.

E.g.: $f^! \mathcal{O}_Y = \mathcal{O}_X$, $\iota: \{y\} \hookrightarrow Y \Rightarrow \iota^! M = M_y$.

$f_*: D(X) \longrightarrow D(Y)$

$M \longmapsto f_\cdot(f^! D_Y \otimes_{D_X} M)$ — note $f^! D_Y \in (f^{-1}D_Y, D_X)$-Mod

$j: U \hookrightarrow Y$ open embedding. $j_* M = j_\cdot(D_U \otimes_{D_U} M) = j_\cdot M$.

$i: \{y\} \hookrightarrow Y$, $i_*(\mathbb{C}) = \iota_\cdot(\underset{\substack{\| \\ D_{Y,y}}}{i^! D_Y} \otimes_{\mathbb{C}} \mathbb{C}) = \iota_\cdot D_{Y,y}$.

④ $j: \mathbb{C}^\times \overset{j}{\hookrightarrow} \mathbb{C} \overset{i}{\hookleftarrow} \{0\}$.

$j_* j^! \mathcal{O}_{\mathbb{C}} = j_* \mathcal{O}_{\mathbb{C}^\times} = j_\cdot \mathcal{O}_{\mathbb{C}^\times} = \mathbb{C}[z, z^{-1}] \circlearrowleft z, \partial_z$ act as usual.

$\iota_* \iota^! \mathcal{O}_{\mathbb{C}} = \iota_* \mathbb{C} = \iota_\cdot D_{\mathbb{C},0} = \mathbb{C}[x, \partial_x] \otimes_{\mathbb{C}[x]} \mathbb{C}$

$\cong \mathbb{C}[\partial_x] \circlearrowleft -z$ acts by lowering

Really: cohom. deg shift there

$= z^{-1}\mathbb{C}[z^{-1}]$ $-\partial_z$ acts by raising...

In general, excision:

Prop: Let $j: U = X \setminus Z \overset{\text{open}}{\hookrightarrow} X \overset{\text{closed}}{\hookleftarrow} Z : i$

Then $\forall\, M \in D(X)$

$$\iota_* \iota^! M \longrightarrow M \longrightarrow j_\cdot j^! M \quad \text{an exact } \Delta.$$

$$\Leftrightarrow \quad M \longrightarrow j_\cdot j^! M \longrightarrow \iota_* \iota^! M[1]$$

① $X=\mathbb{P}^1$ $\quad U=\mathbb{C} \xhookrightarrow{j} \mathbb{P}^1 \xhookleftarrow{i} \{\infty\}$

$\mathcal{O}_{\mathbb{P}^1} \in D(X)$ $\quad \Gamma(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1})=\mathbb{C}$ $\quad (w=z^{-1})$

$j_*j^!\mathcal{O}_{\mathbb{P}^1}=j_*\mathcal{O}_\mathbb{C}$ $\quad \Gamma(\mathbb{P}^1, j_*\mathcal{O}_\mathbb{C})=\Gamma(\mathbb{C},\mathcal{O}_\mathbb{C})=\mathbb{C}[z]=\mathbb{C}[w^{-1}]$

$i_*i^!\mathcal{O}_{\mathbb{P}^1}=i_*\mathbb{C}=\delta_\infty$ $\quad \Gamma(\mathbb{P}^1,\delta_\infty)=$ "sing. types at ∞" $=w^{-1}\mathbb{C}[w^{-1}]$

$\circlearrowright$

$\mathfrak{sl}_2 \simeq \Gamma(\mathbb{P}^1, T_{\mathbb{P}^1}) \hookrightarrow \Gamma(\mathbb{P}^1, D_{\mathbb{P}^1})$

$h = 2w\partial_w$ $\quad$ ($w=x/y$, $y=-1$ chart) to relate to earlier.

$e = w^2\partial_w$

$f = -\partial_w$.

$\mathbb{C}$ ← global fcs.

$\downarrow$

$\mathbb{C}[w^{-1}]$ ← fcs singular at ∞.

$\downarrow$

$w^{-1}\mathbb{C}[w^{-1}]$ ← sing. types

Exercise: Check these satisfy correct. rel^{n}s.

$e = w^2\partial_w$, $\quad f = -\partial_w$.

$h=2w\partial_w$ weight			
0	$\mathbb{C}_{w^0}$	$\mathbb{C}_{w^0}$	0
-2	0	$\mathbb{C}_{w^{-1}}$	$\mathbb{C}_{w^{-1}}$
-4	0	$\mathbb{C}_{w^{-2}}$	$\mathbb{C}_{w^{-2}}$
-6	0	$\mathbb{C}_{w^{-3}}$	$\mathbb{C}_{w^{-3}}$
	$\parallel$	$\parallel$	$\parallel$
	$L(0)$	$M(0)^\vee$	$M(-2)^\vee$

$$L(0) \hookrightarrow M(0)^\vee \twoheadrightarrow M(-2)^\vee$$
$$\mathcal{O}_{\mathbb{P}^1} \hookrightarrow j_*\mathcal{O}_\mathbb{C} \to \delta_\infty$$

$\chi_0 : Z(\mathfrak{g}) \to \mathbb{C}$.. $\quad \mathfrak{sl}_2\mathbb{C} \hookrightarrow \Gamma(\mathbb{P}^1, D_{\mathbb{P}^1})$.

Turns out.

$$U(\mathfrak{sl}_2\mathbb{C})\cdot\ker(\chi_0) \hookrightarrow U(\mathfrak{sl}_2\mathbb{C}) \twoheadrightarrow \Gamma(\mathbb{P}^1,D_{\mathbb{P}^1}).$$

$$h^2+ef+fe \longmapsto 4w\partial_w w\partial_w - \underbrace{w^2\partial_w^2}_{2[w\partial_w w\partial_w - w\partial_w]} - \underbrace{\partial_w w^2\partial_w}_{2[w\partial_w w\partial_w + w\partial_w]} = 0.$$

$$\Rightarrow \quad U(\mathfrak{sl}_2\mathbb{C})_0 \xrightarrow{\simeq} \Gamma(\mathbb{P}^1, D_{\mathbb{P}^1}).$$

← D-Affineness.

Moreover $\Gamma : D(\mathbb{P}^1) \xrightarrow{\simeq} \Gamma(\mathbb{P}^1,D_{\mathbb{P}^1})\text{-Mod} \simeq U(\mathfrak{sl}_2)_0\text{-Mod}$.

To get more general central characters, twisted D-Modules.

$\mathcal{L}$ a line bundle, $D^\mathcal{L} = \mathrm{Diff}(\mathcal{L},\mathcal{L})$ sheaf of algs. plays role of $\mathcal{O}$ ⇝ $\mathcal{L} \in D^\mathcal{L}(X)$ = sheaves of modules over $D^\mathcal{L}$.

General B.B.: Let G ss. alg. group, B a borel, $X=G/B$.

$N \hookrightarrow B \twoheadrightarrow T \xrightarrow{\lambda} \mathbb{C}^\times$ $\quad \lambda$ a dominant coweight.

Then $\mathcal{L}_\lambda = (G\times\mathbb{C}^\times)/B \twoheadrightarrow X$ a $\mathbb{C}^\times$-bundle ⇝ line bundle

Thm [Borel-Weil] $H^0(\mathcal{L}_\lambda) \simeq L(\lambda)$ the simple.

Thm: B.B. $\quad G \circlearrowright X \Rightarrow \mathfrak{g} \to \Gamma(X,T_X)$.

$$D^\lambda(X) \xrightarrow{\Gamma} \Gamma(X, D^\lambda_X)\text{-Mod} \longrightarrow U(\mathfrak{g})\text{-Mod}_\lambda.$$

is an equivalence.

$X = \bigsqcup_{w\in W} X^w$ $\quad X^w \xhookrightarrow{j^w} X$ of dim^n $\ell(w)$.

$$M(w\cdot\lambda)^\vee = \Gamma(X, j^w_* j^{w!}\mathcal{L}^\lambda)$$

and BGG resolution is cons^n wrt Bruhat order.

$$0 \to M(w_0\cdot\lambda) \to \dots \to \bigoplus_{\ell(w)=k} M(w\cdot\lambda) \to \dots \to M(\lambda) \to L(\lambda) \to 0$$

dR cohomology, dR functors

Let $\pi : X \to pt$, $\underset{dR}{\pi_*} : D(X) \longrightarrow Vect$

$M \longmapsto Hom_{D_X}(\mathcal{O}_X, M)$.

$\cdots \to D_X \otimes \wedge^2 \Theta_X \to D_X \otimes \Theta_X \xrightarrow{m} D_X] \simeq D \otimes Sym^\bullet(\Theta[1]) \quad \simeq \text{"} \omega_X \overset{!}{\underset{D_X}{\otimes}} M \text{"}$

$\downarrow$

$0 \to \mathcal{O}_X$

Then $dR(M) = Hom_{D(X)}(D_X \otimes_{\mathcal{O}_X} Sym^\bullet(\Theta[1]), M)$

$\simeq Hom_{\mathcal{O}(X)}(Sym^\bullet(\Theta[1]), M)$.

$\simeq \Gamma(X, \Omega^\bullet_X \otimes_{\mathcal{O}_X} M)$

Note: $D(X) \longrightarrow Sh_{\mathbb{C}}(X) \xrightarrow{\Gamma} Vect$, with dR the composite, where $\Omega^\bullet_X \otimes_{\mathcal{O}_X} M = dR$ ($\Omega^\bullet_X \otimes M$: dR).

E.g. $\mathcal{O}_X \longmapsto \Omega^\bullet_X \simeq \mathbb{C}_X$.

$D_X \longmapsto \mathcal{O}_X$

$D_X \otimes_{\mathcal{O}_X} N = {}_D N \longmapsto N$

"$\ker(P)$" $= H^0[{}_D N \xrightarrow{P} {}_D M] \longmapsto \{n \in N \mid P(n) = 0\}$.

$\iota_* \mathbb{C} = \delta_x \longmapsto \iota_* \mathbb{C} = \mathbb{C}_x$ "skyscraper".

Note:

Def^n: $\mathcal{F} \in Sh^{con}_{\mathbb{C}}(X)$ constructible w.r.t. $X = \coprod_\alpha X^\alpha$ strat.

if $\forall \alpha$ $(j^{\alpha *})(\mathcal{F})$ is "loc. const." on X^α $\forall \alpha$.

(loc. const. cohom.)

Q: When is $dR(M)$ constructible. (i.e. constr. cohom.)

A: ① holonomic ② regular singularities : "sol^{n}s are at most poly^l at ∞"

Recall the filtration $\mathcal{O}_X \hookrightarrow \mathcal{O}_X + T_X \hookrightarrow D_{X \le 2} \hookrightarrow \cdots$ of D_X

$\mathcal{O}_X = D_{X \le 0}$, $\mathcal{O}_X + T_X = D_{X \le 1}$

w. $gr_\bullet D_X = \mathcal{O}(T^*X)$. $\Rightarrow$ $D(X) \longrightarrow QC(T^*X)$.

PBW: $gr_\bullet D_X \leftrightarrow D_X$

Full dim^l $D_X \longmapsto \mathcal{O}_{T^*X}$

holonomic. $\{$ half dim^l $\{$ $\mathcal{O}_X \longmapsto z_* \mathcal{O}_X$ $\quad z : X \hookrightarrow T^*X$.

$\delta_x \longmapsto \iota_* \mathcal{O}_{T^*_x X}$ $\quad \iota : T^*_x X \hookrightarrow T^*X$.

Th^m:

$dR : D_{rh}(X) \xrightarrow{\simeq} Shv^{con}_{\mathbb{C}}(X)$ an equivalence.

Note: These derived categories are built from abelian categories, or "have t-structures", with heart giving " ".

$D_{rh}(X)^\heartsuit$ = D-modules (not cx's!)

$Shv^{con}_{\mathbb{C}}(X)^\heartsuit$ = Sheaves (" ").

These are not intertwined by dR.

Instead define "perverse t-str": data on category equiv. to this one.

an abelian category $Shv^{per}_{\mathbb{C}}(X) \hookrightarrow Shv^{con}_{\mathbb{C}}(X)$

$\simeq$

w. $\to D^b Shv^{per}_{\mathbb{C}}(X)$

$$\begin{tikzcd} D_{rh}(X)^\heartsuit \arrow[r, "dR"] \arrow[d] & Shv^{per}_{\mathbb{C}}(X) \arrow[d] \\ D_{rh}(X) \arrow[r, "dR"] & Shv^{const}(X) \end{tikzcd}$$