Measuring bases

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UC Davis Algebraic Geometry
The question

We would like to compare two bases in representations of $G = GL_m$. 
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- the **MV basis**
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- the MV basis indexed by varieties $Z$
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- *the MV basis* indexed by varieties $Z$
- *the dual semicanonical basis*
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We would like to compare two bases in representations of \( G = GL_m \):

- the MV basis indexed by varieties \( Z \)
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Both bases are crystal bases,
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crystal
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The question

We would like to compare two bases in representations of \( G = GL_m \)

- the \textit{MV basis} indexed by varieties \( Z \)
- the \textit{dual semicanonical basis} indexed by varieties \( Y \)

Both bases are crystal bases, with common polytope models:

\[
\begin{align*}
\text{crystal} & \quad \xrightarrow{b} \\
\text{variety} & \quad \xrightarrow{\text{Pol}} \\
\text{polytope} & \quad \xrightarrow{b} 
\end{align*}
\]

such that \( \text{Pol}(Z) = \text{Pol}(Y) \) whenever \( b(Z) = b(Y) \)

\( KK: \) \textit{if} \( \text{Pol}(Z) = \text{Pol}(Y) \) \textit{do associated basis vectors agree...in some sense?} \)
The answer

No, we have an example.
The answer

No, we have an example.

But first

1. *in what sense?*
No, we have an example.

But first

1. *in what sense?*
2. *tools (equivariant invariants) used to compare*
Roadmap

1. Recollections
2. Setting up the comparison
3. Means to compute
4. Conclusion
Recollections
Let $G$ be an ADE group.
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Write $\Pi = \{\alpha_i\}$ for its simple roots,
Notation

Let $G$ be an ADE group. Fix a reduced expression for the longest word in its Weyl group. (The quantities we’ll consider will depend on it.)

Denote by $U$ a maximal unipotent subgroup. Denote by $\mathcal{U}$ the universal enveloping algebra of its Lie algebra.

Write $\Pi = \{\alpha_i\}$ for its simple roots, and $e_i = e_{\alpha_i}$ for associated Chevalley generators of $\mathcal{U}$. 
Double the simply laced Dynkin quiver of $G$ and denote the associated preprojective algebra by $\mathcal{A}$.
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Denote by $\Lambda$ Lusztig’s nilpotent variety of $\mathcal{A}$-module structures on $\Pi$-graded vector spaces.
Dual semicanonical basis

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Given $Y \in \text{Irr} \Lambda$ let $M \in Y$ be a general point.
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Given $Y \in \text{Irr } \Lambda$ let $M \in Y$ be a general point.

Denote the perfect pairing $\mathcal{U} \times \mathbb{C}[\mathcal{U}] \to \mathbb{C}$ that sends $(a,f)$ to $a \cdot f(1)$ by $\langle a,f \rangle$. 
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If $\dim M = p$ then $f_Y \in \mathbb{C}[\mathcal{U}]$ is defined by the system

$$\langle e_i, f_Y \rangle = \chi(F_i(M)) \text{ for all } i = (i_1, \ldots, i_p)$$
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where $e_{i} = e_{i_1} e_{i_2} \cdots e_{i_p}$ and

$$F_{i}(M) = \{ 0 = M^0 \subset M^1 \subset \cdots \subset M^p = M : M^k / M^{k-1} \cong S_{i_k} \}$$
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Denote the perfect pairing $U \times \mathbb{C}[U] \to \mathbb{C}$ that sends $(a, f)$ to $a \cdot f(1)$ by $\langle a, f \rangle$.

If $\dim M = p$ then $f_Y \in \mathbb{C}[U]$ is defined by the system

$$\langle e_{\hat{i}}, f_Y \rangle = \chi(F_{\hat{i}}(M)) \text{ for all } \hat{i} = (i_1, \ldots, i_p)$$

where $e_{\hat{i}} = e_{i_1} e_{i_2} \cdots e_{i_p}$ and

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Note, $\mathbb{C}[U]$ is graded by the positive root cone.
$G = SL_3 \mathbb{C}$, $\Pi = \{\alpha_1, \alpha_2\}$, quiver

\[
\Lambda = \bigoplus_{(\nu_1, \nu_2)} \{(x_h, x_{\bar{h}}) \in T^* \text{Hom}(\mathbb{C}^{\nu_1}, \mathbb{C}^{\nu_2}) : x_h x_{\bar{h}} = 0 \text{ and } x_{\bar{h}} x_h = 0\}
\]
Example

\[ G = SL_3 \mathbb{C}, \Pi = \{\alpha_1, \alpha_2\}, \text{quiver } \bullet \xrightarrow{\bar{f}} \bullet \]

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Connected component of \( M \in \Lambda \) having \( \text{grdim } M = \alpha_1 + \alpha_2 \) is

\[ \{(x_h, x_{\bar{h}}) \in \mathbb{C}^2 : x_h x_{\bar{h}} = 0\} = \{x_h = 0\} \sqcup \{x_{\bar{h}} = 0\} \]
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\[ = \{z\} \sqcup \{xy - z\} \text{ in } \mathbb{C}[U] \]
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if \( U = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \right\} \)
Theorem (GLS)
The elements \( \{f_Y\} \) as \( Y \) ranges in \( \text{Irr} \Lambda \) form the dual semicanonical basis, denoted \( \mathcal{B}_\Lambda \).
We give the general definition along with the lattice model valid only in type A.
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We give the general definition along with the lattice model valid only in type $A$. Let $T$ be a maximal torus. Denote $\mathcal{O} = \mathbb{C}[t]$ and $\mathcal{K} = \mathbb{C}(t)$. Given $\mu \in X^\bullet(T)$, write $t^\mu$ for its image in $G(\mathcal{K})$ and $L_\mu$ for its image in $G_{\mathcal{K}} = G(\mathcal{K})/G(\mathcal{O})$. 

$$G_{\mathcal{K}} = G(\mathcal{K})/G(\mathcal{O})$$
The affine Grassmannian

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$$\mathcal{G}r = G(\mathcal{K})/G(\mathcal{O}) \overset{A}{=} \left\{ L \mathrel{\overset{\text{free}}{\subset}} \mathcal{O}^m : tL \subset L \right\}$$
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Example: $L_\mu = \text{Span}_{\mathcal{O}}(e_i t^j : 0 \leq j < \mu_i)$.
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$$Gr = G(\mathcal{K})/G(\mathcal{O})_A = \{ L \subset \mathcal{O}^m : tL \subset L \}$$

Example: $L_\mu = \text{Span}_\mathcal{O}(e_it^j : 0 \leq j < \mu_i)$. Fact: $Gr^T = X^\bullet(T)$
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$$Gr = G(\mathcal{K})/G(\mathcal{O}) \cong \{ L \bigcup_{\text{free}}^{\text{rank } m} \mathcal{O}^m : tL \subseteq L \}$$

Example: $L_\mu = \text{Span}_\mathcal{O}(e_i t^j : 0 \leq j < \mu_i)$. Fact: $Gr^T = X^\bullet(T)$ and other distinguished subsets (needed for the definition of MV cycles and later open subset thereof) are all orbits of fixed points.
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$$Gr^\lambda = G(\mathcal{O})L_\lambda = \{ L \in Gr : t|_{\mathcal{O}^m/L} \text{ has Jordan type } \lambda \}$$

$$Gr_\mu = G_1[t^{-1}]L_\mu = \{ L \in Gr : L = \text{Span}_\mathcal{O}(v_1, \ldots, v_m) \text{ such that } v_j = t^{\mu_j}e_j + \sum p_{ij}e_i \text{ with } \deg p_{ij} < \mu_j \}$$

$$S^\mu_- = U_-(\mathcal{K})L_\mu = \{ L \in Gr_\mu : \dim(\mathcal{O}^k/L \cap \mathcal{O}^k) = \mu_1 + \cdots + \mu_k \}$$
Theorem (MV)

The irreducible components of $\overline{Gr}^\lambda \cap S^\mu$ form a basis of cycles—the MV cycles of coweight $(\lambda, \mu)$—for intersection cohomology of $\overline{Gr}^\lambda \cap S^\mu$. 

Calibrating: Fix a highest weight vector $v \in L(\lambda)$, and use Berenstein and Kazhdan's map $L(\lambda) \rightarrow \mathbb{C}[U]$ to make sense of the MV cycles as a basis in $\mathbb{C}[U]$. Denote this basis of $\mathbb{C}[U]$ by $B_{Gr^\lambda}$. Writing $f_Z$ for the avatar of the cycle $Z$. 

MV basis... in $\mathbb{C}[U]$
Theorem (MV)

The irreducible components of $\overline{Gr}^\lambda \cap S^\mu$ form a basis of cycles—the MV cycles of coweight $(\lambda, \mu)$—for intersection cohomology of $\overline{Gr}^\lambda \cap S^\mu$ making it isomorphic to $L(\lambda)_{\mu}$ in representations of the Langlands dual group.

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Calibrating: Fix a highest weight vector \( v_\lambda \in L(\lambda) \), and use Berenstein and Kazhdan’s map \( L(\lambda) \to \mathbb{C}[U] \)

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f_\lambda(u) = v_\lambda^*(u \cdot v)
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Calibrating: Fix a highest weight vector $v_\lambda \in L(\lambda)$, and use Berenstein and Kazhdan’s map $L(\lambda) \to \mathbb{C}[U]$

\[ f_v(u) = v^*_\lambda(u \cdot v) \]

to make sense of the MV cycles as a basis in $\mathbb{C}[U]$. 
Theorem (MV)

The irreducible components of $\overline{Gr}^{\lambda} \cap S^\mu_-$ form a basis of cycles—the MV cycles of coweight $(\lambda, \mu)$—for intersection cohomology of $\overline{Gr}^{\lambda} \cap S^\mu_-$ making it isomorphic to $L(\lambda) \mu$ in representations of the Langlands dual group.

Calibrating: Fix a highest weight vector $v_\lambda \in L(\lambda)$, and use Berenstein and Kazhdan’s map $L(\lambda) \to \mathbb{C}[U]

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to make sense of the MV cycles as a basis in $\mathbb{C}[U]$. Denote this basis of $\mathbb{C}[u]$ by $B_{Gr}$ writing $f_Z$ for the avatar of the cycle $Z$. 
Setting up the comparison
Comparing $\mathcal{B}_{gr}$ and $\mathcal{B}_\Lambda$

We can now compare!

$$\text{Pol}(Y) = \text{Pol}(Z) \Rightarrow f_Y = f_Z$$
Comparing $\mathcal{B}_{gr}$ and $\mathcal{B}_\Lambda$

We can now compare!

$$\text{Pol}(Y) = \text{Pol}(Z) \implies f_Y = f_Z$$

Consider the following invariant.

$$f \in \mathbb{C}[U]_{-\nu} \mapsto \overline{D}(f) = \sum_{i \in \text{Seq}(\nu)} \langle e_i, f \rangle \overline{D}_i \in \mathbb{C}[t^{\text{reg}}]$$

where

$$\overline{D}_i = \prod_{k=1}^{p} \frac{1}{\alpha_{i_1} + \cdots + \alpha_{i_k}} \quad p = \sum \nu_i$$
Reinterpreting $\bar{D}$

Let $f \in \mathbb{C}[U]_{-\nu}$

In case $f = f_Y$

$$\bar{D}(f) = \sum_i \chi(F_i(M))D_i$$

for $M \in Y$ general.
Reinterpreting $\overline{D}$

Let $f \in \mathbb{C}[U]_{-\nu}$

In case $f = f_Y$

$$\overline{D}(f) = \sum_i \chi(F_i(M))\overline{D}_i$$

for $M \in Y$ general.

In case $f = f_Z$

$$\overline{D}(f) = \varepsilon_{L_{-\nu}}(Z)$$

the equivariant multiplicity of $Z$ at its lowest fixed point.
Reinterpreting $\overline{D}$

Let $f \in \mathbb{C}[U]_{-\nu}$

In case $f = f_Y$

$$\overline{D}(f) = \sum_{i} \chi(F_i(M)) \overline{D}_i$$

for $M \in Y$ general.

In case $f = f_Z$

$$\overline{D}(f) = \varepsilon_{L_{-\nu}}(Z)$$

the equivariant multiplicity of $Z$ at its lowest fixed point.

We direct you to the Baumann, Kamnitzer and Knutson paper for explanations. Esp. the appendix.
BKK had a guess as to a pair \((Y, Z)\) in type \(A\) such that \(\text{Pol}(Y) = \text{Pol}(Z)\) but \(f_Z \neq f_Y\).
BKK had a guess as to a pair \((Y, Z)\) in type A such that \(\text{Pol}(Y) = \text{Pol}(Z)\) but \(f_Z \neq f_Y\) and we verified it by checking \(\overline{D}(f_Z) \neq \overline{D}(f_Y)\).
Example

BKK had a guess as to a pair \((Y, Z)\) in type \(A\) such that \(\text{Pol}(Y) = \text{Pol}(Z)\) but \(f_Z \neq f_Y\) and we verified it by checking \(\overline{D}(f_Z) \neq \overline{D}(f_Y)\).

To compute \(\overline{D}(f_Z)\) we need coordinates.
Example

BKK had a guess as to a pair \((Y, Z)\) in type \(A\) such that \(\text{Pol}(Y) = \text{Pol}(Z)\) but \(f_Z \neq f_Y\) and we verified it by checking \(\overline{D}(f_Z) \neq \overline{D}(f_Y)\).

To compute \(\overline{D}(f_Z)\) we need coordinates. For these we relied on the Mirković–Vybornov isomorphism, and our decomposition.
Means to compute
In type $A$, where coweights can be viewed as partitions, the MV isomorphism says that

$$Gr^\lambda \cap Gr_\mu \cong \mathcal{O}_\lambda \cap T_\mu$$
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$$\overline{\text{Gr}}^\lambda \cap \text{Gr}_\mu \cong \overline{\text{O}}_\lambda \cap T_\mu$$

where $\overline{\text{O}}_\lambda$ is the conjugacy class of $J_\lambda$. 
Mirković–Vybornov isomorphism

In type $A$, where coweights can be viewed as partitions, the MV isomorphism says that

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where $\overline{O}_\lambda$ is the conjugacy class of $J_\lambda$ and $\mathbb{T}_\mu$ is the MV slice through $J_\mu$. 
Mirković–Vybornov isomorphism

In type A, where coweights can be viewed as partitions, the MV isomorphism says that

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where $\overline{O}_\lambda$ is the conjugacy class of $J_\lambda$ and $T_\mu$ is the MV slice through $J_\mu$.

We showed that this isomorphism restricts to

$$\phi : \overline{Gr^\lambda} \cap S^\mu_\mu \to \overline{O}_\lambda \cap T_\mu \cap n$$
Mirković–Vybornov isomorphism

In type $A$, where coweights can be viewed as partitions, the MV isomorphism says that

$$\overline{Gr}^\lambda \cap Gr_\mu \cong \overline{O}_\lambda \cap T_\mu$$

where $O_\lambda$ is the conjugacy class of $J_\lambda$ and $T_\mu$ is the MV slice through $J_\mu$.

We showed that this isomorphism restricts to

$$\phi : \overline{Gr}^\lambda \cap S^\mu_- \to \overline{O}_\lambda \cap T_\mu \cap n$$

and that the RHS has decomposition

$$\bigsqcup_{\tau \in S(\lambda)_\mu} X_\tau \quad X_\tau = \hat{X}_\tau^{\text{top}}$$

where

$$\hat{X}_\tau = \{ A \in T_\mu \cap n : A_{|\lambda(i)}| \in O_{\lambda(i)} \text{ for } 1 \leq i \leq m \}$$
Example

Let \( \tau = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \) so \( m = 4 \) and \( r = 2 \). Then \( A \in \mathcal{X}_\tau \) takes the form

\[
\begin{bmatrix}
0 & a & b & c \\
0 & 0 & 0 & e \\
0 & 0 & 0 & f \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
Example

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$ae + bf = 0$
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Recall that both $B_{gr}$ and $B_{A}$ have polytopes—the same polytopes! This means that any other model for the crystal which is equivalent to the polytope model will also be common to both bases.
Interesting in its own right, the above decomposition turns out to reveal a use for the tableaux model of the abstract crystal.

Recall that both $B_{gr}$ and $B_A$ have polytopes—the same polytopes! This means that any other model for the crystal which is equivalent to the polytope model will also be common to both bases. In particular, they have the same tableaux!
Interesting in its own right, the above decomposition turns out to reveal a use for the tableaux model of the abstract crystal. Recall that both $\mathcal{B}_{Gr}$ and $\mathcal{B}_{\Lambda}$ have polytopes—the same polytopes! This means that any other model for the crystal which is equivalent to the polytope model will also be common to both bases. In particular, they have the same tableaux! To every $Y, Z$, one can assign $\tau(Y), \tau(Z)$ and $\tau(Y) = \tau(Z)$ iff $\text{Pol}(Z) = \text{Pol}(Z)$. 

Given an MV cycle $Z$ we showed that we could use its tableau $= (Z)$ to locate the generalized orbital variety that gets sent to an open subset of $\phi(X)$. 

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$$\tau = \tau(Z) \Rightarrow \phi(X_\tau) \subset Z$$
Conclusion
The generalized orbital varieties quickly get quite complicated, and the ideal of the one needed for BKK’s example was unwieldy! But we persevered.
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Using the fact that

$$\varepsilon_p(X) = \frac{\text{mdeg}_W(\lambda_p)}{\text{mdeg}_W(W)}$$

and simply running \texttt{multidegree} in Macaulay2 we found that...
Let \((Y, Z)\) be such that

\[
\tau(Y) = \tau(Z) = \begin{pmatrix}
1 & 1 & 5 & 5 \\
2 & 2 & 6 & 6 \\
3 & 3 \\
4 & 4 
\end{pmatrix}
\]
Counterexample

Let \((Y, Z)\) be such that

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then \(X_\tau\) is 16 dimensional generated in degrees 1, 2, 3, and 6, while a general point \(M\) of \(Y\) is looks like

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 2 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\end{array}
\oplus
\begin{array}{cccc}
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Moreover

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\sum_i \chi(F_i M) D_i \neq \frac{\text{mdeg}_n(X_\tau)}{\text{mdeg}_n(0)}
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Thank you
\[ \overline{D}(Z) = \overline{D}(Q) - 2\overline{D}(P) \]
Example

\[ G = SL_3 \mathbb{C}, \ w_0 = s_1 s_2 s_1, \ \text{and} \ U = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \right\}, \ \text{so that} \ \mathbb{C}[U] = \mathbb{C}[x, y, z] \]
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\( \alpha_1 + \alpha_2 \)

\( \alpha_1 \) in \( \mathfrak{t}_\mathbb{R}^* \cong \mathbb{R}^2 \)
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\[ \alpha_1 \; \text{in} \; t^*_\mathbb{R} \cong \mathbb{R}^2 \; \text{will have “measure”} \; \frac{1}{\alpha_1(\alpha_1 + \alpha_2)} \]
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\( \alpha_1 \) in \( t_\mathbb{R}^* \cong \mathbb{R}^2 \) will have “measure” \( \frac{1}{\alpha_1(\alpha_1 + \alpha_2)} \)

\( \overline{Gr}^{\omega_2} \cong \mathbb{P}^2 \) in \( Gr \)
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\[ \alpha_1 + \alpha_2 \quad \triangleleft \alpha_1 \quad \text{in } t^*_\mathbb{R} \cong \mathbb{R}^2 \text{ will have "measure" } \frac{1}{\alpha_1(\alpha_1 + \alpha_2)} \]

\[ Gr^{\omega_2} \cong \mathbb{P}^2 \text{ in } Gr \text{ and the component of } P(\omega_2) = 1 \rightarrow 2 \text{ in } \Lambda \]
Example

$G = \text{SL}_3 \mathbb{C}$, $w_0 = s_1 s_2 s_1$, and $U = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \right\}$, so that $\mathbb{C}[U] = \mathbb{C}[x, y, z]$

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$\alpha_1 + \alpha_2$

$\alpha_1$ in $t_\mathbb{R}^* \cong \mathbb{R}^2$ will have “measure” $\frac{1}{\alpha_1(\alpha_1 + \alpha_2)}$

$\overline{Gr}_2 \cong \mathbb{P}^2$ in $Gr$ and the component of $P(\omega_2) = 1 \rightarrow 2$ in $\Lambda$ will both correspond to $z \in \mathbb{C}[U]$