Mathematics
UNIVERSITY OF TORONTO

Measuring bases

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UC Davis Algebraic Geometry

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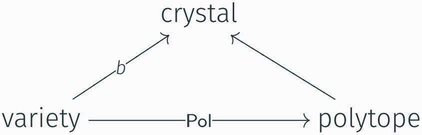

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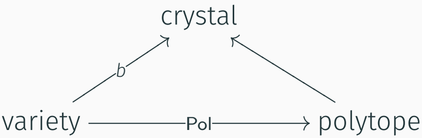

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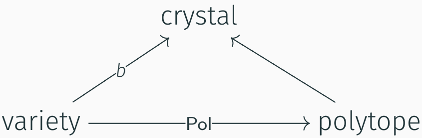

such that $\mathrm{Pol}(Z) = \mathrm{Pol}(Y)$ whenever $b(Z) = b(Y)$

KK: *if* $\mathrm{Pol}(Z) = \mathrm{Pol}(Y)$ *do associated basis vectors agree...in some sense?*

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1. *in what sense?*
2. *tools (equivariant invariants) used to compare*

Roadmap

1. Recollections
2. Setting up the comparison
3. Means to compute
4. Conclusion

Recollections

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Write $\Pi = \{\alpha_i\}$ for its simple roots, and $e_i = e_{\alpha_i}$ for associated Chevalley generators of $\mathcal{U}$.

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Note, $\mathbb{C}[U]$ is graded by the positive root cone.

Example

$G = SL_3\mathbb{C}$, $\Pi = \{\alpha_1, \alpha_2\}$, quiver $\bullet \underset{h}{\overset{\bar{h}}{\leftrightarrows}} \bullet$

$$\Lambda = \bigoplus_{(\nu_1,\nu_2)} \{(x_h, x_{\bar{h}}) \in T^* \operatorname{Hom}(\mathbb{C}^{\nu_1}, \mathbb{C}^{\nu_2}) : x_h x_{\bar{h}} = 0 \text{ and } x_{\bar{h}} x_h = 0\}$$

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if $U = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \right\}$

Dual semicanonical basis in $\mathbb{C}[U]$

Theorem (GLS)

The elements $\{f_Y\}$ as Y ranges in $\operatorname{Irr}\Lambda$ form the dual semicanonical basis, denoted $\mathcal{B}_\Lambda$.

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$$\begin{aligned}
\mathcal{G}r^\lambda = G(\mathcal{O})L_\lambda \quad &= \{L \in \mathcal{G}r : t\big|_{\mathcal{O}^m/L} \text{ has Jordan type } \lambda\} \\
\mathcal{G}r_\mu = G_1[t^{-1}]L_\mu \quad &= \{L \in \mathcal{G}r : L = \mathsf{Span}_{\mathcal{O}}(v_1, \ldots, v_m) \text{ such that} \\
&\qquad v_j = t^{\mu_j} e_j + \sum p_{ij} e_i \text{ with } \deg p_{ij} < \mu_j\} \\
S_-^\mu = U_-(\mathcal{K})L_\mu \quad &= \{L \in \mathcal{G}r_\mu : \dim(\mathcal{O}^k/L \cap \mathcal{O}^k) = \mu_1 + \cdots + \mu_k\}
\end{aligned}$$

MV basis... in $\mathbb{C}[U]$

Theorem (MV)

The irreducible components of $\overline{\mathcal{G}r^\lambda} \cap S^\mu_-$ form a basis of cycles—the MV cycles of coweight (λ, μ)—for intersection cohomology of $\overline{\mathcal{G}r^\lambda} \cap S^\mu_-$

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$$f_v(u) = v_\lambda^*(u \cdot v)$$

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to make sense of the MV cycles as a basis in $\mathbb{C}[U]$. Denote this basis of $\mathbb{C}[u]$ by $\mathcal{B}_{\mathcal{G}r}$ writing f_Z for the avatar of the cycle Z.

Setting up the comparison

Comparing $\mathcal{B}_{\mathcal{G}r}$ and $\mathcal{B}_\Lambda$

We can now compare!

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Consider the following invariant.

$$f \in \mathbb{C}[U]_{-\nu} \mapsto \overline{D}(f) = \sum_{\underline{\mathrm{i}} \in \mathrm{Seq}(\nu)} \langle e_{\underline{\mathrm{i}}}, f \rangle \overline{D}_{\underline{\mathrm{i}}} \in \mathbb{C}[\mathfrak{t}^{\mathrm{reg}}]$$

where

$$\overline{D}_{\underline{\mathrm{i}}} = \prod_{k=1}^{p} \frac{1}{\alpha_{i_1} + \cdots + \alpha_{i_k}} \qquad p = \sum \nu_i$$

Reinterpeting $\overline{D}$

Let $f \in \mathbb{C}[U]_{-\nu}$

In case $f = f_Y$

$$\overline{D}(f) = \sum_{\underline{\mathrm{i}}} \chi(F_{\underline{\mathrm{i}}}(M))\overline{D}_{\underline{\mathrm{i}}}$$

for $M \in Y$ general.

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the equivariant multiplicity of Z at its lowest fixed point.

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We direct you to the Baumann, Kamnitzer and Knutson paper for explanations. Esp. the appendix.

Example

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To compute $\overline{D}(f_Z)$ we need coordinates. For these we relied on the Mirković–Vybornov isomorphism, and our decomposition.

Means to compute

Mirković–Vybornov isomorphism

In type A, where coweights can be viewed as partitions, the MV isomorphism says that

$$\overline{\mathcal{G}r^{\lambda}} \cap \mathcal{G}r_{\mu} \cong \overline{\mathbb{O}}_{\lambda} \cap \mathbb{T}_{\mu}$$

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We showed that this isomorphism restricts to

$$\phi : \overline{\mathcal{G}r^{\lambda}} \cap S^{\mu}_{-} \to \overline{\mathbb{O}}_{\lambda} \cap \mathbb{T}_{\mu} \cap \mathfrak{n}$$

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We showed that this isomorphism restricts to

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and that the RHS has decomposition

$$\sqcup_{\tau \in S(\lambda)_\mu} X_\tau \qquad X_\tau = \overline{\mathring{X}_\tau^{\text{top}}}$$

where

$$\mathring{X}_\tau = \left\{ A \in \mathbb{T}_\mu \cap \mathfrak{n} : A_{|\lambda^{(i)}|} \in \mathbb{O}_{\lambda^{(i)}} \text{ for } 1 \leq i \leq m \right\}$$

Example

Let $\tau = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$ so $m = 4$ and $r = 2$. Then $A \in \mathring{X}_\tau$ takes the form

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$$\tau = \tau(Z) \Rightarrow \phi(X_\tau) \subset Z$$

Conclusion

Coordinates on generalized orbital varieties

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Using the fact that

$$\varepsilon_p(X) = \frac{\mathrm{mdeg}_W(\mathring{X}_p)}{\mathrm{mdeg}_W(W)}$$

and simply running `multidegree` in Macaulay2 we found that...

Counterexample

Let (Y, Z) be such that

$$\tau(Y) = \tau(Z) = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 5 & 5 \\ \hline 2 & 2 & 6 & 6 \\ \hline 3 & 3 \\ \cline{1-2} 4 & 4 \\ \cline{1-2}\end{array}$$

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Thank you

Backup slides

$$\overline{D}(Z) = \overline{D}(Q) - 2\overline{D}(P)$$

Example

$G = SL_3\mathbb{C}$, $w_0 = s_1 s_2 s_1$, and $U = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \right\}$, so that $\mathbb{C}[U] = \mathbb{C}[x, y, z]$

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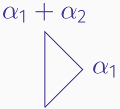

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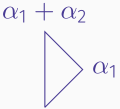
1	2
3	



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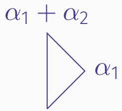

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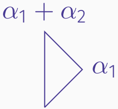

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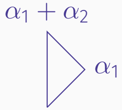

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