Rep th: $X^n \xrightarrow[\text{proper}]{f} Y$ $\qquad A = H^{BM}_{2n}(X \times_Y X)$

Ex: X, Y finite sets $\qquad$ Ans: A = block diagonal matrices.

Claim: $A = \mathrm{End}(f_* \underline{\mathbb{C}}_X[n])$

Thm $f : \underset{\text{smooth}}{X^n} \to Y$ proper (not allowed hopf fibr or $\mathbb{C}^\times \hookrightarrow \mathbb{C}^2$...)

Then $f_* \underline{\mathbb{C}}_X[n] = \bigoplus_i (\text{semisimple perverse sheaves})[i]$

$$= \bigoplus_i \bigoplus_{\chi} IC_\chi[i].$$

Def $X \xrightarrow[\text{proper}]{f} Y$ is semismall if for each stratum of Y

$\dim(\text{stratum}) + 2\dim(\text{fibre}) \leq \dim X = \dim Y$ (assume)

those strata for which this ineq is strict are called relevant.

($\Rightarrow$ generically f is 1-to-1 since the locus where the fibres are ≥ 1 dim is codim ≥ 2 ...)

Ex: f is semismall $\iff f_* \underline{\mathbb{C}}[n]$ is perverse

$$\iff \dim X \times_Y X \leq n.$$

Thm f proper semismall $\Rightarrow$ no shifts in the decomp thm

Only relevant strata contribute

i.e. $f_* \underline{\mathbb{C}}[n] = \bigoplus_{S_k \text{ rel. stratum}} IC_{\bar{S}_k}(L_k) = \bigoplus_{\substack{S_k \text{ rel stratum} \\ x \in \widehat{\pi_1(S_k)}}} IC_{\bar{S}_k}(L_k^x) = \bigoplus IC_{\bar{S}_k}(L^x) \otimes \underbrace{V_k^x}_{\text{multiplicity of } L^x}$

The fibre of this local system at $y_k \in S_k$ is $H^{\cdot}_{top}(f^{-1}(y_k))$.

$= \mathbb{C}\left\langle \begin{matrix} \text{irred comp} \\ \text{of } f^{-1}(y_k) \end{matrix} \right\rangle$

With this we have that the algebra A which we were interested in also decomposes

$$\mathrm{End}(f_* \underline{\mathbb{C}}[n]) = \bigoplus \mathrm{End}(IC_{\bar{S}_k}(L_k)) \underset{\text{Schur's lem.}}{=} \bigoplus \mathrm{End}(V_k^x)$$

thus we see it manifestly as a semisimple algebra generated by components of $X \times_Y X$.

Q! why?

Eg: $IC_{S^1}(L_1)^{\otimes 2} = IC_{S^1}(L_1) \otimes \mathbb{C}^2$

Springer theory takes $X \to Y = \tilde{\mathcal{N}} \to \mathcal{N}$.

$$\begin{tikzcd} T^*\mathcal{B} \cong \tilde{\mathcal{N}} \arrow[r, hook] \arrow[d, "p"] & \tilde{\mathfrak{g}} \arrow[rr] \arrow[d, "\mu"] & & \mathfrak{t}^* \arrow[d] \\ \chi^{-1}(0) =: \mathcal{N} \arrow[r, hook, "i"] & \mathfrak{g}^* \arrow[r, "\chi"] & \mathfrak{g}^*/\!/G \cong & \mathfrak{t}^*/\!/W \end{tikzcd}$$

p is proj to the cotangent fibre; 1-to-1 over regular elts.

Symplectic aside

$T^*\mathcal{B} = T^*G/\!/B$ ← ham red

$B = TU$, $= T^*(G/U)/\!/T$ ← quot + m.m

$(T^*G)/\!/U = T^*(G/U)$

$\Rightarrow \tilde{\mathfrak{g}} = T^*(G/U)/T$

$= \{(\mathfrak{b}, X) : X \in \mathfrak{b}\}$

$e \in \mathcal{N}$; $p^{-1}(e) =: \mathcal{B}_e$ "Springer fibre"

Rmk $\mu^{-1}(e) =: \tilde{\mathcal{B}}_e$ has the same underlying space as $\mathcal{B}_e$ but w/ non-reduced scheme str.

To see this we restrict maps to regular semisimple elts and find that while p is 1-to-1, μ is W-to-1.:

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$$\begin{tikzcd} \tilde{\mathfrak{g}}^{rs} \arrow[r] \arrow[d, "\mu^{rs}"', "W \text{ cover}"] & \mathfrak{t}^{*,reg} \arrow[d] \\ \mathfrak{g}^{*,rs} \arrow[r] & \mathfrak{t}^{*,reg}/\!/W \end{tikzcd} \implies \text{different scheme str!}$$

Eg $\cdot\ e = 0 \implies \mathcal{B}_e = \{\mathfrak{b} \in \mathcal{B} : 0 \in \mathfrak{b}^\perp\} = \mathcal{B}$.

$\cdot\ e = J_{(n)} \in \mathfrak{sl}(n) \implies \mathcal{B}_e = \{pt\}$.

Ex $\tilde{\mathcal{B}}_e = \operatorname{Spec} H^\cdot(\mathcal{B})$ if $e = J_{(n)}$.

Eg (cont'd) $\cdot\ e = \begin{bmatrix} 0 & 0 & 1 \\ & 0 & 1 \\ & & 0 \end{bmatrix}$ subregular. $\mathcal{B}_e = \{ l \subseteq H \subseteq \mathbb{C}^3 : e\mathbb{C}^3 \subseteq H,\ eH \subseteq l,\ el = 0 \}$

$n=3$

Ex. $= \mathbb{P}^1 \sqcup_{pt} \mathbb{P}^1$

Ex $\mathcal{B}_e$ for e subregular = Dynkin configuration $\mathbb{P}^1$'s.

Q! Why?

Ex $G = SL(4)$, $e = \begin{bmatrix} 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}$. $\mathcal{B}_e = \{ \overbrace{\underbrace{l}_{\mathbb{P}^1}}^{\mathbb{P}^1} \subseteq \overbrace{\underbrace{P}_{\mathbb{P}^1}}^{e_1 e_2} \subseteq \overbrace{\underbrace{H}_{pt}}^{\mathbb{P}^1} \subseteq \mathbb{C}^4 : \cdots \}$.

Eg / Ex $e = \begin{bmatrix} 0 & 0 & 1 & 0 \\ & 0 & 1 & \\ & & 0 & \\ & & & 0 \end{bmatrix}$ $\mathcal{B}_e = \mathcal{O}_{\mathbb{P}^1}(2)$

$\mathbb{P}^1$ choices for l, P either $e_1 e_2$ or not

i.e. $\mathbb{P}(\mathbb{C}^4/\langle e_1, e_2\rangle) = \mathbb{P}(\mathbb{C}^2) = \mathbb{P}^1$

Thm $H^{BM}_{2n}(\tilde{\mathcal N} \times_{\mathcal N} \tilde{\mathcal N}) = \mathbb{C}[W]$ **Ex** $p: \tilde{\mathcal N} \to \mathcal N$ is a semismall map.

Cor (to Ex.) A is a semisimple alg. generated by irred. comps of the Steinberg var.

Goal Realize W action on $p_* \underline{\mathbb{C}}[\dim \tilde{\mathcal N}] =: S_{\mathcal N}$

To wit Consider $S_{\mathfrak g} := \mu_* \underline{\mathbb{C}}[\dim \tilde{\mathfrak g}]$. It's related to $S_{\mathcal N}$.

Facts 1] $i^* S_{\mathfrak g} = S_{\mathcal N}[\dim \mathfrak t]$. (base change... which is ok because topologically

$$\begin{array}{ccc} \tilde{\mathfrak g} & \to & \tilde{\mathcal N} \\ \downarrow & \ulcorner & \downarrow \\ \mathfrak g & \hookleftarrow & \mathcal N \end{array}$$
)

2] $S_{\mathfrak g}$ and $S_{\mathcal N}$ are Fourier transforms of each other up to shifts.

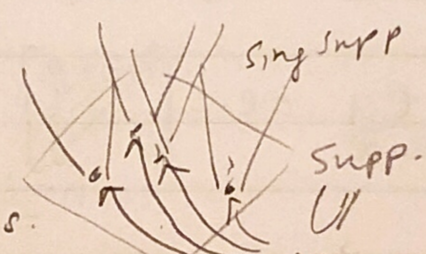
Reason $S_{\mathfrak g} \in Sh(\mathfrak g^*) \xrightarrow[\sim]{FT} Sh(\mathfrak g)$

more generally FT identifies $Sh(V)$ and $Sh(V^*)$.

$SS(S_{\mathfrak g}) \subseteq T^*(\mathfrak g^*) \cong \mathfrak g^* \times \mathfrak g$.

"the sing supp is in the fibre dir"

Fast and loose.



Note Fibres of χ are $\underbrace{\text{reg. ss.}}_{\text{Stab = max torus}}$ conj classes.

$$\left\{ \begin{array}{l} \lambda \in \mathfrak t^*/\!/W \\ \chi^{-1}(\lambda) =: \mathcal O_\lambda \cong G/T = T^*(G/B) \text{ twisted} \end{array} \right\} = \text{family of smoothings. (cf. resolving)}$$

$$G/T \xrightarrow{\ \upsilon\ } G/B$$

3] $\Psi(\underline{\mathbb{C}}_{\mathfrak g^{rs}}[\dim \mathfrak g]) = S_{\mathcal N} \circlearrowleft \pi_1(\mathfrak t^{*,reg}/\!/W) = \underbrace{B_W}_{\text{Braid grp}}$

Ψ: nearby cycles functor.

Ex $\tilde{\mathfrak g} \xrightarrow{\mu} \mathfrak g^*$ is small. i.e. no rel. strata of positive codim.
(a smart const. of the W action)

Corr $\mu_* \underline{\mathbb{C}}[\dim \mathfrak g] = IC(\mathcal L_W) \circlearrowleft W$ where $\mathcal L_W$ is the loc syst μ^{rs}.

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Refs
- Chris' ε7/1/04
- Nadler?

$A \curvearrowright H^{\cdot}(\mathcal{B}_e)$

Thm Every irrep of $\mathbb{C}[W]$ appears as an action on $H^{\cdot}(\mathcal{B}_e)$.