To prove Theorem 3 it is necessary to consider the adjoint functor $G: C(S) \to C(\Lambda)$. It is defined as follows: $G(\mathbf{W}) = \mathbf{V} = \mathrm{Hom}_k(\Lambda, \mathbf{W})$; $\partial(v)\lambda = -\Sigma x_i v(\xi_i \lambda) + d(v(\lambda))$; $V_j^i(\Lambda_k) \subset W_{j+k}^{i-j-k}$. Although the image $G(C^b(S))$ does not lie in $C^b(\Lambda)$, G allows one to define a functor $G_D: D^b(S) \to D^b(\Lambda)$. Using the Koszul complex, it is easy to verify that the functor $G_D$ is inverse to the function $F_D$.

5. Let $\mathscr{F}, \mathscr{I}$ be the full subcategories in $D^b(S)$ and $D^b(\Lambda)$, generated by the complexes, consisting of finite-dimensional (respectively free) modules. It is easy to verify that $F_D^{-1}(\mathscr{F}) = \mathscr{I}$, so that $F_D$ defines an equivalence of categories $D^b(\Lambda)/\mathscr{I} \to D^b(S)/\mathscr{F}$ (the quotient categories in the sense of Verdier [7]).

Using Serre's theorem, describing the category Coh in terms of $\mathscr{M}(S)$ (see [9]), it is easy to get that the category $D^b(\mathrm{Coh})$ is equivalent with $D^b(S)/\mathscr{F}$. Thus, from Theorem 3 follows

THEOREM 4. The categories $D^b(\mathrm{Coh})$ and $D^b(\Lambda)/\mathscr{I}$ are equivalent.

6. Proposition. The natural imbedding $\mathscr{M}(\Lambda) \to D^b(\Lambda)$ defines an equivalence of categories $\mathscr{M}(\Lambda)/\mathscr{P} \to D^b(\Lambda)/\mathscr{I}$.

The proposition follows from the fact that free $\Lambda$-modules are projective and injective. Theorem 2 follows from this proposition and Theorem 4.

7. Theorems 1-4 are true for any field k; Theorems 3 and 4 are true if k is replaced by an arbitrary basis Z, $\Xi$ by a locally free sheaf of $\mathscr{O}_Z$-modules, P by a projective spectrum of sheaves of algebras S = S(X), where $X = \Xi^*$.

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# COHERENT SHEAVES ON $P^n$ AND PROBLEMS OF LINEAR ALGEBRA


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The goal of this note is to generalize the results of Horrocks and Barth [1], and Drinfel'd and Manin [2] to the case of projective space of any dimension n. In particular, for any coherent sheaf L on $\mathbf{P}^n$ there will be constructed a "two-sided resolution" which is unique up to homotopy (a complex $K^\cdot$ with $H^0(K^\cdot) = L$, $H^i(K^\cdot) = 0$ for $i \neq 0$), the i-th term of which is isomorphic with $\bigoplus_j H^{i+j}(\mathbf{P}^n, L(-j)) \otimes \Omega^j(j)$ (generalized "monads" of Barth). The precise formulation of the result uses the derived categories of Verdier [3].

1. Let C be a triangulated category. We shall say that a family of its objects $\{X_i\}$ generates C, if the smallest full triangulated subcategory containing them is equivalent with C.

LEMMA 1. Let C and D be triangulated categories, F: C → D be an exact functor, $\{X_i\}$ be a family of objects of C. Let us assume that $\{X_i\}$ generates C, $\{F(X_i)\}$ generates D, and for any pair $X_i$, $X_j$ from the family F: $\mathrm{Hom}^\cdot(X_i, X_j) \to \mathrm{Hom}^\cdot(F(X_i), F(X_j))$ is an isomorphism. Then F is an equivalence of categories.

2. Let $A^\cdot$ be a graded algebra. Notation: $A^\cdot[i]$ is the free one-dimensional graded $A^\cdot$-module with distinguished generator of degree i; $M_{[0,n]}(A^\cdot)$ is the full subcategory of


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the category of graded A -modules and morphisms of degree 0, whose objects are isomorphic with finite direct sums of $A^\cdot[i]$, where $0 \leqslant i \leqslant n$; $K^b_{[0,n]}(A^\cdot)$ is the category whose objects are finite complexes over $M_{[0,n]}(A^\cdot)$, and whose morphisms are morphisms of complexes modulo null-homotopic ones.

3. With an (n + 1)-dimensional vector space V over the field k are associated two graded algebras $\Lambda^\cdot(V^*)$ and $S^\cdot(V)$. We set $K_\Lambda = K^b_{[0,n]}(\Lambda^\cdot(V^*))$, $K_S = K^b_{[0,n]}(S^\cdot(V))$. (It is clear that replacing $\Lambda^\cdot(V^*)$ by $\Lambda^\cdot(V^*)/\Lambda^{N+1}(V^*)$ and $S^\cdot(V)$ by $S^\cdot(V)/(S^{n+1}(V))$ gives an equivalent category $K^b_{[0,n]}$.) Let P be n-dimensional projective space over k, $V = H^0(P, O(1))$.

LEMMA 2. For any pair i, j such that $0 \leqslant i, j \leqslant n$, and $l \geqslant 1$

$$\mathrm{Hom}\,(\Omega^i(i), \Omega^j(j)) = \Lambda^{i-j}(V^*), \qquad \mathrm{Ext}^l(\Omega^i(i), \Omega^j(j)) = 0,$$
$$\mathrm{Hom}\,(O(-i), O(-j)) = S^{i-j}(V), \qquad \mathrm{Ext}^l(O(-i), O(-j)) = 0,$$

where composition of morphisms coincides with multiplication in $\Lambda^\cdot(V^*)$ and $S^\cdot(V)$, respectively.

The lemma is proved by induction with the aid of the exact sequence $0 \to \Omega^i(i) \to \Lambda^i(V) \otimes O \to \Omega^{i-1}(i) \to 0$.

4. Let M(P) be the category of coherent sheaves on P, and $D^b(P)$ be its derived category. It follows from Lemma 2 that there exist natural additive functors $\widetilde{F}_1: M_{[0,n]}(\Lambda^\cdot(V^*)) \to M(\mathbf{P})$ and $\widetilde{F}_2: M_{[0,n]}(S^\cdot(V)) \to M(\mathbf{P})$ such that

$$\widetilde{F}_1(\Lambda^\cdot(V^*)[i]) = \Omega^i(i), \quad \widetilde{F}_2(S^\cdot(V)[i]) = O(-i).$$

They extend canonically to exact functors $F_1: K_\Lambda \to D^b(\mathbf{P})$, $F_2: K_S \to D^b(\mathbf{P})$.

THEOREM. $F_1$ and $F_2$ are equivalences of categories.

Proof. We verify that $F_i$ satisfy Lemma 1 (by $\{X_i\}$ is meant $\{\Lambda^\cdot(V^*)[i]\}$ and $\{S^\cdot(V)[i]\}$, respectively, $0 \leqslant i \leqslant n$). According to Lemma 2, it suffices to show that $\Omega^i(i)$ (respectively, $O(-i)$, $0 \leqslant i \leqslant n$) generate $D^b(P)$. On P × P the Koszul resolution of the diagonal Δ has the form

$$0 \to p_1^*(\Omega^n(n)) \otimes p_2^*(O(-n)) \to \ldots \to p_1^*(\Omega^1(1)) \otimes p_2^*(O(-1)) \to O_{\mathbf{P}\times\mathbf{P}} \to O_\Delta \to 0$$

(here $p_i: \mathbf{P} \times \mathbf{P} \to \mathbf{P}$ are the projections). In particular, $O_\Delta$, which means also any object of $D^b(\mathbf{P} \times \mathbf{P})$ of the form $O_\Delta \overset{L}{\otimes} Lp_2^*(X)$, where $X \in Ob D^b(\mathbf{P})$, belongs to the full triangulated subcategory of $D^b(\mathbf{P} \times \mathbf{P})$, generated by sheaves of the form $p_1^*(\Omega^i(i)) \otimes p_2^*(Y)$. It is evident from the projection formulas that $X = Rp_{1*}(O_\Delta \overset{L}{\otimes} Lp_2^*(X))$ belongs to the full triangulated subcategory of $D^b(P)$, generated by $\Omega^i(i) \otimes Rp_{1*}(p_2^*(Y)) = \Omega^i(i) \underset{k}{\otimes} R\Gamma(Y)$, i.e., generated by $\Omega^i(i)$. Consideration of the formula $X = Rp_{2*}(Lp_1^* \overset{L}{\otimes} O_\Delta)$ gives the result for $O(-i)$ which was required.

Remarks. 1. The Koszul resolution makes it possible to construct explicitly functors inverse to $F_i$.

2. In terms of $K_\Lambda$ and $K_S$ it is easy to calculate functors of type $\overset{L}{\otimes}$ and R Hom on $D^b(P)$. For example, the functor on $K_\Lambda$, corresponding to the tensor product on $D^b(P)$, is defined as follows. An object $F_1^{-1}(\Omega^i(i) \otimes \Omega^j(j))$ is represented by a complex in degrees 0 and 1:

$$\bigoplus_{j \leqslant m \geqslant i} \Lambda^{i+j-m}(V) \underset{k}{\otimes} \Lambda^\cdot(V^*)[m] \to \bigoplus_{j > l < i} \Lambda^{i+j-l}(V) \underset{k}{\otimes} \Lambda^\cdot(V^*)[l].$$

Here the differential is induced by the exterior product under the isomorphism

$$\mathrm{Hom}_{\Lambda^\cdot(V^*)}(\Lambda^{i+j-m}(V) \underset{k}{\otimes} \Lambda^\cdot(V^*)[m], \; \Lambda^{i+j-l}(V) \underset{k}{\otimes} \Lambda^\cdot(V^*)[l]) = \mathrm{Hom}_k(\Lambda^{i+j-m}(V) \otimes \Lambda^{m-l}(V), \Lambda^{i+j-l}(V)).$$

3. The theorem is easily generated to the case of projective bundles over any base (whence follows directly a known theorem about $K_0$).

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# DEGREE OF DEGENERACY OF A SINGULAR POINT OF A VECTOR FIELD ON THE PLANE

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In families of vector fields on the plane, depending on a finite number of parameters, one can encounter degenerate singular points. A degenerate singular point with given phase portrait can occur for certain values of the parameters stably with respect to small motions of the family: in any close family, for close values of the parameters there occurs a singular point with the original phase portrait.

In studying the suspension of the phase portrait in the family it is important to know the minimal number of parameters for which a singular point "of given form" can occur stably. Here the concept of "given form" can be made precise in various ways; we shall understand by it a singular point of given $C^r$-orbital type.

The following results allow one to calculate this number for $C^r$-sufficient jets of a vector field at a singular point. In this paper this is called the $C^r$-codimension.

Notation. We denote by $V^r(\varepsilon_2)$ the Lie algebra (ring) of germs at the point $(0) \in \mathbf{R}^2$ of vector fields of class $C^r$ (functions of class $C^\infty$) on the plane ($r = 1, 2, \ldots, \infty$; in the case $r = \infty$ the index r is omitted).

By $J^kV$ we denote the space of k-jets of germs from V, by $\pi_k$ we denote the natural projection $\pi_k$: $V \to J^kV$.

## 1. $C^r$-codimension

1.1. Definition. We call germs $u, v \in V$, $C^r$-orbitally equivalent ($u \overset{r}{\sim} v$), if the phase portraits of representatives of the germs u and v are carried into one another (preserving the direction of motion along phase curves) with the aid of the germ at the point $(0) \in \mathbf{R}^2$ of a diffeomorphism of class $C^r$ $\psi: \mathbf{R}^2 \to \mathbf{R}^2$, $\psi(0) = 0$.

1.2. Definition. We call a k-jet $q \in J^kV$ $C^r$-sufficient if $\forall\, u, v \in V$: $\pi_k u = \pi_k v = q \Rightarrow u \overset{r}{\sim} v$.

1.3. Definition. We call $C^r$-sufficient k-jets $q, p \in J^kV$ $C^r$-orbitally equivalent, $q \overset{r}{\sim} p$, if $\exists u, v \in V$: $\pi_k u = q$, $\pi_k v = p$, $u \overset{r}{\sim} v$.

1.4. Notation. By $\mathcal{O}^r(q)$, where $q \in J^kV$ is a $C^r$-sufficient k-jet, we denote the set of $C^r$-sufficient k-jets $p \in J^kV : p \overset{r}{\sim} q$.

LEMMA (see [3]). $\mathcal{O}^r(q)$ is a semialgebraic submanifold of the space $J^kV$.

1.5. Definition. The $C^r$-codimension of a $C^r$-sufficient k-jet $q \in J^kV$ will mean the codimension of the submanifold $\mathcal{O}^r(q)$ in the space $J^kV$.

## 2. Dual Objects

2.1. Notation. By $\Lambda_m$ (respectively, $\Lambda_m^*$), m = 0, 1, 2, we denote the space of m-vector fields of class $C^\infty$ on the plane, which decrease rapidly at infinity (respectively, the space dual to $\Lambda_m$ formed by exterior m-forms on the plane with coefficients in the algebra of slowly growing distributions on the plane).

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