Sturm-Liouville Theory

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Abstract

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A basic introduction into Sturm-Liouville Theory. We mostly deal with the general 2nd-order ODE in self-adjoint form. There are a number of things covered including: basic asymptotics, properties of the spectrum, interlacing of zeros, transformation arguments...

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Chapter 1

The Basic Theory

1.1 A Brief Review of ODE

To start things off, lets get some notation and definitions under our belt.

Notation. An open interval is denoted by (a, b) with $-\infty \leq a < b \leq \infty$; $[a, b]$ denotes the closed interval which includes the left endpoint a and right endpoint b, regardless of whether these are finite or infinite. $\mathbb{R}$ denotes the reals, $\mathbb{C}$ the complex numbers, $\mathbb{N}$ the natural numbers without zero, $\mathbb{N}_0$ the natural numbers with zero, and $\mathbb{Z}$ as the integers. For any interval $J \subseteq \mathbb{R}$, $L^1(J, \mathbb{C})$ denotes the linear manifold of complex valued Lebesgue measurable function $y : J \to \mathbb{C}$ such that

$$\int_a^b |y(x)|dx \equiv \int_J |y(x)|dx \equiv \int_J |y| < \infty$$

$L^1_{loc}(J, \mathbb{C})$ is used to denote the linear manifold of functions $y \in L^1([\alpha, \beta], \mathbb{C})$ for all compact intervals $[\alpha, \beta] \subseteq J$. Note if $J = [a, b]$ and a, b are both finite, then $L^1_{loc}(J, \mathbb{C}) = L(J, \mathbb{C})$. Also, we denote the collection of complex-valued functions y which are absolutely continuous on all compact intervals $[\alpha, \beta] \subseteq J$ by $AC_{loc}(J)$.

We'll now go over some of the basis theorems and formula's from ODE theory. The first being of course our uniqueness and existence theorem justifying our future analysis of our Sturm-Liouville operators

Theorem 1.1 (Picard-Lindelöf)**.** *Suppose that $f(t)$ is Lipschitz continuous, $C^{0,1}$, on some interval $J \subset \mathbb{R}$. Then the first-order ODE*

$$\begin{cases} y' = f(x, y(x)) \\ y(x_0) = y_0 \end{cases}$$

has a unique solution about (x_0, y_0) at least locally.

Proof. The standard proof is via Picard iterates, by defining $\phi_0 = y_0$ and

$$\phi_k = y_0 + \int_{x_0}^{t} f(s, \phi_{k-1}(s))ds$$

It's easy to show this converges to our solution $y(t)$ with the $C^{0,1}$ assumption. □

Something interesting happens if we rephrase an n-order ODE in terms of a first order matrix system. Let

$$\dot{Y} = AY \iff p_n y^{(n)} + \ldots + p_1 y^{(1)} + p_0 y^{(0)} = 0$$

where Y can be thought as a column vector containing the $y^{(0)}$ to $y^{(n-1)}$, A is an n by n matrix containing the coefficients of the ODE. Note that we can add f and F to gain the correspondence to the non-homogeneous case. Via diagonalization or Jordan diagonalization methods we notice that there are n independent solutions (eigenvectors), namely

$$\dot{Y} = AY \iff \dot{U} = DU \quad or \quad \dot{U} = JU$$

under $\Lambda Y = U$ where $\Lambda \in GL(n, \mathbb{C})$, the matrix of eigenvectors for A which satisfies

$$\Lambda A \Lambda^{-1} = D \quad or \quad \Lambda A \Lambda^{-1} = J$$

Note that D is the diagonal matrix of the eigenvalues of A(i.e. A is not defective) and J is the Jordan matrix of A (i.e. A is defective). Thus the system has n independent solutions.

Definition 1.1 (Fundamental Matrix X)**.** The fundamental matrix $X(\cdot)$ is the collection of all n independent solutions.

$$X(\cdot) = \left(\vec{\lambda_1} y_1(\cdot) \dots \vec{\lambda_n} y_n(\cdot)\right)$$

Notice that X solves

$$\dot{X} = AX$$

Furthermore, notice that as long as $\det(X) \neq 0$, the system has n linearly independent solution (fundamental solutions). This leads us to give this determinate a special name.

Definition 1.2 (Wronskian)**.**

$$W(y_1, \dots, y_n)(\cdot) \equiv \det X(\cdot) = \begin{vmatrix} y_1 & \dots & y_n \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix}$$

This isn't the only formula we have to calculate this value though, we can determine it directly from the system with the following identity.

Theorem 1.2 (Liouville's Formula)**.** *Let X be a fundamental solution of $\dot{X} = AX$ with $X(x_0) = X_0$. Then*

$$\det X(\cdot) = \det X_0 \exp\left(\int_{x_0}^{x} trace\,(A(s))\,ds\right)$$

Proof. Let $X = x_{i,j}$ and $A = a_{i,j}$, and recall the Leibniz formula for determinants:

$$\det(X) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{n=1}^{n} x_{\sigma(n),n}$$

This allows us to calculate the derivative one row at a time. i.e.

$$\det(X)' = \sum_{i=1}^{n} \begin{vmatrix} x_{1,1} & x_{1,2} & \dots & x_{1,n} \\ \vdots & & & \vdots \\ x'_{i,1} & x'_{i,2} & \dots & x'_{i,n} \\ \vdots & & & \vdots \\ x_{n,1} & x_{n,2} & \dots & x_{n,n} \end{vmatrix}$$

Now, since X solves $\dot{X} = AX$, we have

$$x'_{i,k} = \sum_{j=1}^{n} a_{i,j} x_{j,k}, \quad i, k \in \{1, \ldots, n\}$$

Thus, the derivative row can be written as

$$(x'_{i,1}, \ldots, x'_{i,n}) = \sum_{j=1}^{n} a_{i,j}(x_{j,1}, \ldots, x_{j,n}), \quad i \in \{1, \ldots, n\}$$

Recall that subtracting rows from a matrix does not change the determinate, thus if we subtract

$$(x'_{i,1}, \ldots, x'_{i,n}) = \sum_{j=1, j \neq i}^{n} a_{i,j}(x_{j,1}, \ldots, x_{j,n})$$

from the derivative row we obtain

$$\begin{vmatrix} x_{1,1} & x_{1,2} & \ldots & x_{1,n} \\ \vdots & & & \vdots \\ x'_{i,1} & x'_{i,2} & \ldots & x'_{i,n} \\ \vdots & & & \vdots \\ x_{n,1} & x_{n,2} & \ldots & x_{n,n} \end{vmatrix} = \begin{vmatrix} x_{1,1} & x_{1,2} & \ldots & x_{1,n} \\ \vdots & & & \vdots \\ a_{i,i}x_{i,1} & a_{i,i}x_{i,2} & \ldots & a_{i,i}x_{i,n} \\ \vdots & & & \vdots \\ x_{n,1} & x_{n,2} & \ldots & x_{n,n} \end{vmatrix} = a_{i,i} \det X$$

Thus we have the first order O.D.E

$$\det X' = \sum_{i=1}^{n} a_{i,i} \det X = \operatorname{trace}(A) \det X$$

The formula immediately follows. □

Now that we've reviewed the matrix formulation, we have an immediate consequence of the Picard-Lindelöf theorem.

Corollary 1.1. *Suppose that* $A \in M_{n\times n}(L^1_{loc}(J, \mathbb{C}))$ *and* $F \in M_{n\times m}(L^1_{loc}(J, \mathbb{C}))$. *Then the first-order system*

$$\begin{cases} \dot{X} = AX + F \\ X(x_0) = X_0 \end{cases}$$

has a unique solutions on all of J. *Furthermore, if* X_0, A, F *are all real-valued, then then solution is real valued.*

Thus, if $A \in M_{n\times n}(L^1_{loc}(J))$, we know there is exactly one matrix solution X satisfying $X(x_0) = 1$, where 1 denotes the $n \times n$ identity matrix.

Definition 1.3 (Primary Fundamental Matrix Φ)**.** For each fixed $x_0 \in J$, let $\Phi(\cdot, x_0)$ be the fundamental matrix solution satisfying

$$\begin{cases} \dot{\Phi} = A\Phi \\ \Phi(x_0, x_0) = 1 \end{cases}$$

Note that for each fixed x_0 in J, $\Phi(\cdot, x_0)$ belongs to $M_{n\times n}(AC_{loc}(J))$. Furthermore, if J is compact and $A \in M_{n\times n}(L^1(J, \mathbb{C}))$, then $\Phi(\cdot, x_0) \in M_{n\times n}(AC(J))$. We also have that $\Phi(x, x_0)$ is invertible for each $x, x_0 \in J$ and

$$\Phi(x, x_0) = X(x)X^{-1}(x_0)$$

for any fundamental matrix X. You can think of $\Phi(x, x_0)$ as the exponential function, namely $\Phi(x, x_0) = \exp(A(x - x_0))$ by defining the exponential of a matrix as Taylor series. Now for another important formula from ODE theory.

Theorem 1.3 (Variation of Parameters Formula)**.** *. Let J be any interval, and suppose our data is $L^1_{loc}(J, \mathbb{C})$ and let Φ be the primary fundamental matrix of the first order system*

$$\begin{cases} \dot{X} = AX + F \\ X(x_0) = X_0 \end{cases}$$

Then we have that

$$X(x) = \Phi(x, x_0)X_0 + \int_{x_0}^{x} \Phi(x, s)F(s)ds, \quad x \in J$$

is the matrix solution to the first order system.

Proof. Clearly $X(x_0) = X_0$. Differentiating the formula gives the result. □

The last thing we mention is dependance on the initial data. Before we do so, we'll need the Gronwall Inequality

Lemma 1.4 (The Gronwall Inequality). *Assume $g \in L^1(J, \mathbb{R})$ with $g \geq 0$ a.e. and f real valued and continuous on J. If y is continuous, real valued, and satisfies*

$$y(x) \leq f(t) + \int_a^t g(s)y(s)ds, \quad \forall t \in J$$

then

$$y(t) \leq f(t) + \left(\int_a^t f(s)g(s) \exp \left(\int_s^t g(u)du \right) du \right), \quad \forall t \in J$$

Proof. Let $h(t) = \int_a^t gy$ where $t \in J$ and notice

$$h' = gy \leq g(f+h) \quad a.e. \iff h' - gh \leq gf \quad a.e.$$

along with the equality of

$$\exp\left(-\int_a^s g(u)du\right)\left(h'(s) - g(s)h(s)\right) = \frac{d}{ds} \exp\left(-\int_a^s g(u)du\right) h(s)$$

Therefore, integrating from a to t we obtain

$$\exp\left(-\int_a^t g(u)du\right) h(t) \leq \int_a^t g(s)f(s) \exp\left(-\int_a^s g(u)du\right) ds, \quad t \in J$$

Now if we put it all together with the main assumption, we see that

$$\begin{aligned} y(t) \leq & f(t) + h(t) \leq f(t) + \exp\left(\int_a^t g(u)du\right) \int_a^t g(s)f(s) \exp\left(-\int_a^s g(u)du\right) ds \\ \leq & f(t) + \int_a^t g(s)f(s) \exp\left(\int_s^t g(u)du\right) ds, \quad t \in J \end{aligned}$$

□

Note that this is sometimes called the left Gronwall inequality since we started from the left endpoint a. Similarly, we have the right Gronwall inequality

$$y(t) \leq f(t) + \left(\int_t^b f(s)g(s) \exp \left(\int_t^s g(u)du \right) du \right), \quad \forall t \in J$$

if

$$y(t) \leq f(t) + \int_t^b g(s)y(s)ds, \quad \forall t \in J$$

The proof is almost identical.

Lemma 1.5. *Let* $x_0, z_0 \in J, X_0, Z_0 \in M_{n\times m}(\mathbb{C}), A, B \in M_{n\times n}(L^1(J,\mathbb{C})), F, G \in M_{n\times m}(L^1(J,\mathbb{C}))$*. Assume*

$$\begin{cases} \dot{X} = AX + F \\ X(x_0) = X_0 \end{cases} \quad \& \quad \begin{cases} \dot{Z} = BZ + G \\ Z(z_0) = Z_0 \end{cases}$$

Then

$$|X(t) - Z(t)| \le K \exp\left(\int_J |B|\right), \quad \forall t \in J$$

where

$$K = |X_0 - Z_0| + \left|\int_{x_0}^{z_0} |F|\right| + M\left|\int_{x_0}^{z_0} |A|\right| + \int_J |F - G| + M\int_J |A - B|$$

and

$$M = \left(|X_0| + \int_J |F|\right) \exp\left(\int_J |A|\right)$$

Proof. It follows from the Gronwall inequality in the interior of J by considering the integral forms

$$X(t) = X_0 + \int_{x_0}^{t} \big(A(s)X(s) + F(s)\big)ds \quad \& \quad Z(t) = Z_0 + \int_{y_0}^{t} \big(B(s)Z(s) + G(s)\big)ds$$

The endpoints follow as well by considering some strictly increasing sequence that converges to them. The sequence turns the solution into Cauchy sequence. □

Theorem 1.4 (Continuous Data Dependance)**.** *Let* $x_0 \in J, X_0 \in M_{n\times m}(\mathbb{C}), A \in M_{n\times n}(L^1(J,\mathbb{C}))$*, and* $F \in M_{n\times m}(L^1(J,\mathbb{C}))$*. Let* $X = X(\cdot, x_0, X_0, A, F)$ *be the solution to*

$$\begin{cases} \dot{X} = AX + F \\ X(x_0) = X_0 \end{cases}$$

Then X *is a continuous function in all of its variables* x_0, X_0, A, F *uniformly on the closure of* J*; more precisely, for fixed* A, F, x_0, X_0 *and given any* $\epsilon > 0$*, there is a* $\delta > 0$ *such that if* $y_0 \in J, Z_0 \in M_{n\times m}(\mathbb{C}), B \in M_{n\times n}(L^1(J,\mathbb{C}))$*, and* $G \in M_{n\times m}(L^1(J,\mathbb{C}))$ *satisfy*

$$|x_0 - y_0| + |X_0 - Z_0| + \int_J |A - B| + \int_J |F - G| < \delta$$

then

$$|X(t, x_0, X_0, A, F) - X(t, z_0, Z_0, B, G)| < \epsilon, \quad \forall t \in J$$

Note that $Y(t, \cdot)$ is jointly continuous in x_0, X_0, A, F, uniformly for t in the closure of J.

Proof. The absolute continuity of the Lebesgue integral and our δ bound imply that the constant K in the previous lemma can be made arbitrarily small. The claim follows. □

Now that about wraps up a quick review of some of the major ideas from ODE. Let's move onto our main topic of consideration!

1.2 The Sturm-Liouville Equation

In the 1800's, Jacques Charles Francois Sturm [1803-1855] and Joseph Liouville [1809-1882] worked on a particular second-order linear differential operator, namely

$$\mathcal{L}[y(x)] = \frac{1}{\omega}\left(-\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y(x)\right)$$

It is also considered in it's (non)-homogeneous form differential equation form:

$$-\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y(x) = f(x)$$

Due to the work started by Sturm and Liouville, the equation is rightfully named the Sturm-Liouville Differential Equation. The problem was essentially solving the eigenvalue problem for the differential operator $\mathcal{L}$. The weight ω naturally applies to the spectrum in this form. This problem shows up all over the place with weaker and weaker assumptions and it is for that reason it is still a very active area of research. Over the course of these notes we'll cover tools and machinery needed to tackle this problem under a variety of assumptions, namely on the coefficients of the above equation. We'll begin with an nice example from Quantum Mechanics to showcase it's importance.

Example 1.6 (The Hydrogen Atom)**.** Erwin Schrödinger[1887-1961] thought of an equation to model the state of particles that are comparable on a Planck scale ($\hbar$), namely

$$\hat{H}\Phi = \left(-\frac{\hbar^2}{2m}\Delta + V(x,t)\right)\Phi(x,t) = i\hbar\frac{\partial}{\partial t}\Phi(x,t)$$

where m is the mass of the particle and $V(x,t)$ was the potential of the system. The striking similarity with the Classic Hamiltonian is immediate in this form, thus it seems very plausible that this equation should give analogous results in the Quantum setting. In the case of the hydrogen atom, we're working with an electrostatic potential in $\mathbb{R}^3$,

$$V(x,t) = V(|x|) = \frac{1}{4\pi\epsilon_0}\frac{q}{|x|}$$

where $q, \epsilon_0 \in \mathbb{R}$. A common clever trick utilized in PDE, is the assumption of a separable solution, i.e. $\Phi(x,t) = \Psi(x)\psi(t)$. The result of such an assumption is an uncoupling of the multi-parameter equation into single-parameter equations, thus reducing a PDE into a set of ODE's. In this case we obtain:

$$\frac{d\psi}{dt} = -i\lambda\psi(t) \quad \& \quad \left(-\frac{\hbar^2}{2m}\Delta + V(|x|)\right)\Psi(x) = \lambda\Psi(x)$$

where λ is a spectral variable obtained by splitting. The equation on the right is generally known as the time-independent Schrödingier equation, and follows if $V(x,t) = V(x)$ as we have in this case. Since the meat and bones of the problem lies in this second equation, we focus our attention on it. Due to the radial symmetry of the potential, we separate variables again, $\Psi(x) = R(r)Y(\theta,\phi)$. By the same procedure, expanding the spherical Laplacian allows us to separate equations once again:

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{2mr^2}{\hbar^2}\left(V(r) - \lambda\right)R(r) = l(l+1)R(r)$$

$$\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 Y}{\partial\phi^2} = -l(l+1)\sin\theta\, Y(\theta,\phi)$$

Notice the separation constant was chosen to be $l(l+1) \in \mathbb{C}$, to be consistent with the literature. We see that we've reach a point of solving two Sturm-Liouville equations. It turns out that the angular equation has solutions stemming from Legendre polynomials. The radial equation has solutions in the form of spherical Bessel and Neumann functions. Physicists call our eigenvalue, λ, the energy eigenvalue, it's usually denoted E.

The trick we used in the above example for separating the solution is given credit to Joseph Fourier[1768-1830]. His motivation for his work comes from the famous

Example 1.7 (The Wave Equation)**.** Leonhard Euler [1707-1783] stumbled about a model to describe waves in $\mathbb{R}^3$, namely

$$\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0$$

Assuming a separable solution, $u(t, x) = T(t)P(x)$, here provides us with another example of a Sturm-Liouville Problem. Namely the eigenvalue problem of the laplacian:

$$\Delta P = -\lambda P$$

In the previous example we expanded in spherical coordinates, but we can expanded in cartesian coordinates this time to find Fourier's equation:

$$X_i'' = -\lambda_i X_i$$

where $P(x) = X_1 X_2 X_3$ and $\lambda_1 + \lambda_2 + \lambda_3 = \lambda$. This will become our fundamental example of our self-adjoint Sturm-Liouville operator, and we will cover this later.

Let's ask ourself a few questions now. Where does the Sturm-Liouville Equation hold, on an interval or on the whole real line? Where do we want our coefficients to live? What about periodic or anti periodic coefficients? Can we weaken the requirement that y is twice differentiable? We have a number of different avenues ahead of us now and we somewhat see how these are relevant to the real world. In keeping with our notation we explicitly rate everything back to our review formulation. Let

$$A = \begin{pmatrix} 0 & 1/p \\ q & 0 \end{pmatrix} \quad \& \quad F = \begin{pmatrix} 0 \\ f \end{pmatrix} \quad \& \quad Y = \begin{pmatrix} y \\ py' \end{pmatrix}$$

We see that

$$\dot{Y} = AY + F \iff -\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y(x) = f(x)$$

One may notice that in this form we don't require y to be strictly twice differentiable. We simply require that py' is differentiable, as of such, quite a few authors [Ref's] use the following terminology:

Definition 1.8. The quasi-derivative of y is denoted by $y^{[1]}$. It is defined by $y^{[1]} \equiv py'$.

This notation is helpful when working in weaker spaces to get around the differentiability of y'. Now let's talk about where we want our coefficients to live. Our big players will be $1/p, q$, you may wonder why $1/p$ and not just p. Have a look at our reduction to a 1st order system and notice that the entries of A were $1/p$ and q. Thus the Picard-Lindelöf depends on where these coefficients live. Overall this will help us gain a better understanding of when we can formulate the Sturm-Liouville Equation in a variety of forms. From here on in, we'll use the shorthand notation of

$$y' = \frac{dy}{dx} \quad \& \quad (y^{[1]})' = \frac{d}{dx}(py')$$

We consider the three forms of the Homogeneous Sturm-Liouville Equation, $\mathcal{L}_0$. The first form, we'll call the formal self-adjoint form of the Sturm-Liouville Equation

$$-(y^{[1]})' + qy = 0$$

The second form, we'll call the expanded Sturm-Liouville Equation

$$y'' + fy' + gy = 0$$

The third form, we'll call the self-adjoint form of the Sturm-Liouville Equation

$$-y'' + hy = 0$$

Definition 1.9 (Solution)**.** By a solution to the formal self-adjoint Sturm-Liouville Equation we mean a function $y : J \to \mathbb{C}$ such that y and $y^{[1]}$ are absolutely continuous on each compact subinterval of J and the equation is satisfied $a.e$ on J.

Now for a few comments on the above forms, obviously the last form is the nicest and corresponds to $p = 1$ in some sense. There has been quite a bit of work done with the self-adjoint form, but some more recent work has shifted to the formal self-adjoint case due to some weaker assumptions. We'll now quickly show how the formal self-adjoint and the self-adjoint forms are effectively the same under a nice transformation and assumptions. Assume $p > 0$ a.e. on some interval J, i.e. p doesn't change sign on J. If we further assume that $1/p \in L^1(J)$, let

$$t = \int_\xi^x \frac{ds}{p(s)}$$

where $\xi \in J$, i.e. this takes the interval $J \to J'$ and is well defined. By substitution into the formal self-adjoint form we obtain:

$$\begin{aligned}\mathcal{L}_0(y)(t) &= -\frac{d^2y}{dt^2} + p(x)q(x)y(t) = 0\\ &= -\frac{d^2y}{dt^2} + h(t)y(t) = 0\end{aligned}$$

where $h(t) = p(x)q(x)$. Notice that $h \in L^1(J')$ if we assume further that $q \in L^1(J)$ since

$$\int_{J'} |h(t)|dt = \int_J p(x)|q(x)|\frac{dx}{p(x)} = \int_J |q(x)|dx$$

We're starting to see that L^1 is a nice space to work in due to this simplification. What about if we're not as fortunate to have L^1, but L^1_{loc} instead? Luckily we can preform a similar change, namely

$$\tan(\phi) = \int_\xi^x \frac{ds}{p(s)}$$

This avoids the issue of ϕ being infinite since $\arctan t$ lives between $[-\pi/2, \pi/2]$. Furthermore, if we modify y in the following manner

$$z(\phi) = y(x)\cos\phi$$

This produces the following simplification

$$\begin{aligned}\mathcal{L}_0(y)(\phi) &= -\frac{d^2z}{d\phi^2} + \left(1 + p(x)q(x)\sec^4\phi\right) z(\phi) = 0\\ &= -\frac{d^2z}{d\phi^2} + h(\phi)z(\phi) = 0\end{aligned}$$

For our typical condition of $h(\phi) \in L^1$, we see that require a little more from q this time, namely

$$\int_{J'} |h(\phi)| d\phi = \int_J \left|1 + p(x)q(x)(1+t(x)^2)^2\right| \frac{dx}{(1+t(x)^2)p(x)} \lesssim \int_J |q(x)|(1+t(x)^2)dx < \infty$$

As we're now seeing, it is possible to switch between the forms under suitable conditions and transformations. These different assumptions lead to unique areas of Sturm-Liouville Theory, and we'll go over a fair list but definitely not exhaust it.

1.3 Fundamental Solutions

We begin this section with the previously mentioned example of Fourier's Equation:

Example 1.10. Let $\lambda \in \mathbb{C}$ and consider

$$y''(x) = -\lambda^2 y(x), \quad x \in [a,b]$$

Clearly we have the following general solution to the O.D.E

$$y(x) = A\exp(i\lambda x) + B\exp(-i\lambda x), \quad A, B \in \mathbb{R}$$

The solution here corresponds to the Spectral Sturm-Liouville Equation with $q = 0$ and $\lambda \to \lambda^2$ for cosmetics. The "standard" initial data chosen for this problem is

$$\underbrace{y_1(\xi) = 1 \quad \& \quad y_1'(\xi) = 0}_{Dirichlet} \qquad \text{or} \qquad \underbrace{y_2(\xi) = 0 \quad \& \quad y_2'(\xi) = 1}_{Neumann}$$

where $\xi \in [a,b]$. From the general solution, we deduce that

$$y_1(x) = \cos(\lambda(x-\xi)) \quad \& \quad y_2(x) = \frac{\sin(\lambda(x-\xi))}{\lambda}$$

We will show later on that the eigenvalues of the "regular" Sturm-Liouville problem are necessary real, so we can view cases of $q \neq 0$ *a.e.* as perturbations from this simple example.

The non-homogeneous case has a similar nice form via the variation of constants. Namely:

Lemma 1.11. *The general solution to*

$$y''(x) + \lambda^2 y(x) = f(x)$$

is given by

$$y(x) = Ac_\lambda(x) + Bs_\lambda(x) + \int_\xi^x \left(c_\lambda(x)s_\lambda(s) - c_\lambda(s)s_\lambda(x)\right) f(s)ds$$

where

$$c_\lambda(x) = \cos(\lambda(x-\xi)) \quad \& \quad s_\lambda(x) = \frac{\sin(\lambda(x-\xi))}{\lambda}$$

Proof. Apply the variation of parameters formula. □

In general, we don't have a nice formula for $q \neq 0$ but we know there exists a fundamental set of solutions. Thus we define

$$c(x, \lambda, 1/p, q, \omega) \quad \& \quad s(x, \lambda, 1/p, q, \omega)$$

as the fundamental solutions to

$$-(y^{[1]})' + qy = \lambda\omega y$$

satisfying

$$\underbrace{c(\xi, \cdot) = 1, \quad \& \quad c'(\xi, \cdot) = 0}_{Dirichlet} \qquad \& \qquad \underbrace{s(\xi, \cdot) = 0, \quad \& \quad s'(\xi, \cdot) = 1}_{Neumann}$$

This allows us to define the general solution to the Sturm-Liouville Equation as

$$y(x, \cdot) = Ac(x, \cdot) + Bs(x, \cdot), \quad A, B \in \mathbb{C}$$

These fundamental solutions satisfy a nice property, by modifying the Wronskian

Definition 1.12 (Modified Wronskian)**.**

$$W_p(y_1, y_2)(\cdot) \equiv \begin{vmatrix} y_1 & y_2 \\ y_1^{[1]} & y_2^{[1]} \end{vmatrix} = y_1 y_2^{[1]} - y_2 y_1^{[1]}$$

Lemma 1.13. *The Wronskian for the solutions to Sturm-Liouville equation with "standard" initial data are given by*

$$W_p(c,s)(x) = 1$$

Proof. Apply the Liouville formula. □

Theorem 1.5. *Every nontrivial solution $y, y^{[1]}$ of the Sturm-Liouville Equation with data $1/p, q, \omega \in L^1(J,\mathbb{C})$ are entire functions in λ of order at most $1/2$. i.e.*

$$|y(x,\lambda)| \leq C\exp(M\sqrt{|\lambda|}) \quad \& \quad |y^{[1]}(x,\lambda)| \leq C\exp(M\sqrt{|\lambda|})$$

Proof. Notice

$$\frac{d}{dx}\left(|\lambda||y|^2 + |y^{[1]}|^2\right) = |\lambda|\left(\frac{y\overline{y^{[1]}}}{p} + \frac{y^{[1]}\overline{y}}{p}\right) + \overline{y^{[1]}}(q-\lambda\omega)y + y^{[1]}(\overline{q} - \overline{\lambda}\overline{\omega})\overline{y}$$

Using this and the inequality

$$2|ab| \leq \frac{|\lambda||a|^2 + |b|^2}{\sqrt{|\lambda|}}, \quad |\lambda| \neq 0$$

we obtain

$$\frac{d}{dx}\left(|\lambda||y|^2 + |y^{[1]}|^2\right) \leq \frac{|\lambda||y|^2 + |y^{[1]}|^2}{\sqrt{|\lambda|}}\left(\frac{|\lambda|}{|p|} + |q| + |\lambda||\omega|\right)$$

Recalling that $\partial_x \log(f) = f'/f$, we see that

$$\frac{d}{dx}\left(\log(|\lambda||y|^2 + |y^{[1]}|^2)\right) \leq \frac{\sqrt{|\lambda|}}{|p|} + \frac{|q|}{\sqrt{|\lambda|}} + \sqrt{|\lambda|}|\omega|$$

Integrating the above gives the result with

$$M = \int_J \frac{1}{|p|} + |\omega| \quad \& \quad B > \exp\left(\frac{1}{\sqrt{|\lambda|}}\int_J |q|\right)$$

□

1.4 Prüfer Transformation

The Sturm-Liouville Equation is somewhat difficult to work with in the form

$$-(y^{[1]})' + qy = \lambda\omega y$$

Luckily, there exists a nice transformation to go polar coordinates (i.e. a polar factorization). The benefits of this will become apparent in due time, but to give a vague idea it allows us to count zeros in a very efficient manner via an argument principal method by splitting the Sturm-Liouville equation to obtain a first order Θ equation. Consider the following Prüfer equations:

$$\rho' = \left[\left(\frac{1}{p} + \lambda\omega - q\right)\sin\Theta\cos\Theta\right]\rho \qquad \& \qquad \Theta' = (q - \lambda\omega)\cos^2\Theta - \frac{\sin^2\Theta}{p} \tag{1.1}$$

Theorem 1.6. *If $(y, y^{[1]})$ is a solution to the Sturm-Liouville equation and (ρ, Θ) solve the above equations, then we have that $y = \rho\cos\Theta$ and $y^{[1]} = \rho\sin\Theta$.*

Proof. Suppose that $y = \rho\cos\Theta$ and $y^{[1]} = \rho\sin\Theta$, then we have

$$\frac{y^{[1]}}{y} = \tan\Theta \quad \& \quad \rho^2 = y^2 + (y^{[1]})^2$$

Differentiating the first relation gives

$$\frac{\Theta'}{\cos^2\Theta} = \frac{(y^{[1]})'}{y} - \frac{1}{p}\frac{(y^{[1]})^2}{y^2} \iff \Theta' = (q - \lambda\omega)\cos^2\Theta - \frac{\sin^2\Theta}{p}$$

Differentiating the second relation gives

$$\rho\rho' = \frac{yy^{[1]}}{p} + y^{[1]}(y^{[1]})' \iff \rho' = \left[\left(\frac{1}{p} + \lambda\omega - q\right)\sin\Theta\cos\Theta\right]\rho$$

□

Note that there is no fixed form of the transformation, you could choose $y = \rho\sin\Theta$ and $y^{[1]} = \rho\cos\Theta$

$$\Theta' = \frac{\cos^2\Theta}{p} + (\lambda\omega - q)\sin^2\Theta$$

$$\rho' = \left[\left(q - \frac{1}{p} - \lambda\omega\right)\sin\Theta\cos\Theta\right]\rho$$

The differences really just come down to sign choices though. More importantly, notice that Θ gives us a very nice method for counting zero's of y. To outline why this is the case, we'll showcase the properties of Θ in a theorem.

Theorem 1.7. *Suppose Θ solves the Prüfer angle equation with $\Theta(a,\lambda)=\alpha$, $\alpha\in[0,\pi)$. We'll assume $1/p, q, \omega \in L^1(J,\mathbb{R})$ with $p,\omega>0$ a.e. Then the unique solution $\Theta(x,\lambda)$ is defined on J and has the following properties*

- *$\Theta(b,\lambda)$ is continuous and strictly increasing in λ*
- *$\Theta(b,\lambda)\to\infty$ as $\lambda\to\infty$*
- *$\Theta(b,\lambda)\to 0$ as $\lambda\to-\infty$*

Proof. By our earlier ODE review, we know the solution is continuous by our restriction of data. Let $\tan\theta=\sqrt{\lambda}\tan\Theta$ for $\lambda>0$ and require $|\Theta-\theta|<\pi/2$ to determine θ uniquely. Then

$$\theta'=\sqrt{\lambda}\frac{\cos^2\theta}{p}+\frac{1}{\sqrt{\lambda}}(\lambda\omega-q)\sin^2\theta$$

Along with an integration we see

$$\theta(b,\lambda)\geq\theta(a,\lambda)+\sqrt{\lambda}\int_J\min(1/p,\omega)-\frac{1}{\sqrt{\lambda}}\int_J|q|$$

For large λ, the q term clearly dies. We assumed that $\min(1/p,\omega)>0$. Therefore $\phi(b,\lambda)\to\infty$ as $\lambda\to\infty$ which implies the same goes for $\Theta(b,\lambda)$. To prove the last claim, let

$$\lim_{\lambda\to-\infty}\Theta(x,\lambda)=L(x)$$

The limit exists since Θ is strictly increasing in λ. We just need to find what it is, so integrate the original Pr̈fer angle equation to obtain

$$\Theta(b,\lambda)=\alpha+\int_J\frac{\cos^2\Theta}{p}+\int_J(\lambda\omega-q)\sin^2\Theta$$

If we rewrite this equation to bound the ω term, we see that

$$\left|\lambda\int_J\omega\sin^2\Theta\right|\leq|\Theta(b,\lambda)|+\alpha+\int_J\frac{1}{p}+\int_J|q|$$

We have that $\Theta(b,\lambda)$ is bounded for $\lambda<0$, thus we have

$$\int_J\omega\sin^2\Theta\to 0\quad\text{as }\lambda\to-\infty$$

By defining an approximating sequence of function $f_n = \omega \sin^2 \Theta(\cdot, \lambda_n)$ where $\lambda_n \to -\infty$ we can apply Lebesgue dominated Convergence to obtain

$$\omega \sin^2 L(x) = 0 \quad a.e. \implies L(x) = 0 \mod \pi$$

Now we just have to show $L = 0$ $a.e.$, so consider $(\xi, \eta] \subset J$ and integrate on this interval and apply the limit to obtain

$$L(\eta) - L(\xi) \leq \int_\xi^\eta \frac{1}{p}$$

Using the initial value we deduce

$$L(\eta) \leq \alpha + \int_\xi^\eta \frac{1}{p} < \pi$$

For all $\eta \in [a, c]$ for some $c \in J$. Hence $L(\cdot) = 0$ on $(\xi, \eta]$. Let $c = lub\{\eta \in J : L(\eta) = 0\}$ (lower upper bound). By the same argument, we see that $c = b$. Thus $L(\cdot) = 0$ on J. □

Since any zero, $y(x_n, \cdot) = 0$, implies $\Theta(x_n, \cdot) = n\pi + \mathcal{O}(1)$ by continuity where $n \in \mathbb{Z}$. Furthermore, we have that the number of zero's (N) of any given solution $y(x, \lambda, \cdot)$ are given by

$$N(\lambda) = \frac{1}{\pi}\Theta(b, \lambda) + \mathcal{O}(1)$$

where b is the right end point. Thus we see that Prüfer transformations give us the perfect tool to estimate the asymptotics of $N(\lambda)$, and in turn λ_n. Notice that we can find the zeros of $y^{[1]}$ just as easily by using cotangent, i.e $\cot \Theta = y/y^{[1]}$. As we've stated before, we see that the zeros of $y^{[1]}$ lie between the zeros of y.

1.5 Separation and Comparison

As you may imagine, the difficulty of finding an explicit solution for the general case is impossible. Many different Sturm-Liouville Equations have been studied over the years, so we do have some reliable results available to us. Subtle changes may lead to drastically different solutions, so it'd be nice if we had a tool to tell us when solutions to two different Sturm-Liouville Problems are similar in nature (e.g. location of zeros). This section deals with machinery to help distinguish the similarities between equations. To do this, we start with a definition classifying the boundary of our interval J, i.e. a and b.

Definition 1.14 (Regular and Singular Endpoints)**.** For the non-homogeneous Sturm-Liouville, the endpoint a of J is called regular if

$$1/p, q, f \in L^1((a,d),\mathbb{C})$$

for some $d \in J$; otherwise it is called singular. Similarly, the other endpoint b is called regular if

$$1/p, q, f \in L^1((d,b),\mathbb{C})$$

for some $d \in J$.

We now remark that we have continuous dependance of solutions on the Sturm-Liouville problem.

Theorem 1.8 (Continuous Data Dependence)**.** *Let $\xi, \eta \in J$ and $A_i, B_i \in \mathbb{C}$. Let y denote the unique solution to the initial value problem:*

$$\begin{cases} -(y^{[1]})' + qy = f \\ y(\xi) = A_1, y^{[1]}(\xi) = B_1 \end{cases}$$

with $1/p_i, q_i, f_i \in L^1_{llc}(J,\mathbb{C})$. Then each of y and $y^{[1]}$ are jointly continuous functions in all variables, uniformly on compact subintervals of J. More precisely, given $\epsilon > 0$ and a compact subinterval of J containing η, ξ, there exists a $\delta > 0$ such that if

$$|\eta - \xi| + |A_1 - A_2| + |B_1 - B_2| + \int_K \left(\left| \frac{1}{p_1} - \frac{1}{p_2} \right| + |q_1 - q_2| + |f_1 - f_2| \right) < \delta$$

then

$$|y(x, A_1, B_1, 1/p_1, q_1, f_1) - y(x, A_2, B_2, 1/p_2, q_2, f_2)| < \epsilon$$

and

$$|y^{[1]}(x, A_1, B_1, 1/p_1, q_1, f_1) - y^{[1]}(x, A_2, B_2, 1/p_2, q_2, f_2)| < \epsilon$$

both for all $x \in K$. Furthermore, if $1/p, q, f \in L^1(J, \mathbb{C})$, J is a compact interval, and our δ bound holds with $K = J$ then the above ϵ bounds hold on J.

Proof. This is a consequence of the Continuous Data Dependence from the ODE review section. □

Thus, it makes sense that we can even talk about how "close" to Sturm-Liouville Problems are. To make this meaning, we show the zeros of nontrivial solutions are isolated at all regular points.

Lemma 1.15. *Let $-(y^{[1]})' + qy = 0$ hold with real valued p, q such that $p > 0$ a.e on J and λ is real. Then the zeros of a non-trivial solution are isolated in J and are regular at the end points a and b. If the solution has a zero at a regular endpoint, then there is a one sided neighbourhood where it is isolated. Thus only a singular endpoint of J can be an accumulation point of zeros of any nontrivial solution.*

Proof. Without the loss of generality, suppose that a non-trivial solution y has consecutive zeros at $\xi, \eta \in J$ with $\xi < \eta$. This implies that $y^{[1]}(h) = 0$ for some $h \in (\xi, \eta)$ since

$$0 = y(\eta) - y(\xi) = \int_\xi^\eta y' = \int_\xi^\eta \frac{y^{[1]}}{p} = y^{[1]}(h) \int_\xi^\eta \frac{1}{p}$$

by the Mean Value Theorem for Lebesgue integration (since $y^{[1]}$ is continuous on J). The assumption that $p > 0$ a.e. allows us to assert that $y^{[1]}(h) = 0$ for some $h \in (c, d)$. Now suppose there exists some sequence $\{x_n \in J : n \in \mathbb{N}\}$ such that $x_n \to x_0$ and $y(x_n) = 0$ for all $n \in \mathbb{N}$(i.e. not isolated zeros). In-between each of these zero's we know there is some $h_n \in (x_n, x_{n+1})$ such that $y^{[1]}(h_n) = 0$ for all $n \in \mathbb{N}$. However, this implies that $y(x_0) = y^{[1]}(x_0) = 0$ which means that the solution is actually trivial on J by uniqueness of the initial value problem. This is a contradiction, therefore the zeros must be isolated. □

Since we've shown that the zeros are isolated, we present the Sturm Separation Theorem which tells us the zeros of our solutions are intertwined.

Theorem 1.9 (Sturm Separation)**.** *Let* $-(y^{[1]})' + qy = 0$ *hold with real valued* p, q *such that* $p > 0$ *a.e. Suppose that* y_1 *and* y_2 *are linearly independent solutions of the Sturm-Liouville equation, then* y_2 *has a zero strictly between any two zeros of* y_1

Proof. Suppose that $y_1(\xi) = y_1(\eta) = 0$. Since y_1 and y_2 are linearly independent, neither is trivial or have a common zero. Thus we can assume that ξ and η are consecutive zeros of y_1 and $\xi < \eta$. Without the loss of generality assume that $y_1 > 0$ on (ξ, η) (i.e. replace y_1 with $-y_1$ if necessary). Thus

$$0 < y_1(x) - y_1(\xi) = \int_\xi^x \frac{1}{p} y_1^{[1]}, \quad x \in (\xi, \eta)$$

This implies that $y_1^{[1]}(\xi) > 0$ since we assumed $p > 0$. Repeating the argument with the right endpoint η revels that $y_1^{[1]}(\eta) < 0$. Now we perform a clever manipulation to the ODE.

$$\begin{cases} -y_2(y_1^{[1]})' + qy_1y_2 = 0 \\ -y_1(y_2^{[1]})' + qy_1y_2 = 0 \end{cases} \implies 0 = y_2(y_1^{[1]})' - y_1(y_2^{[1]})' = \left[y_1^{[1]}y_2 - y_1y_2^{[1]}\right]'$$

Integrating on (ξ, η) gives

$$0 = y_1^{[1]}(\eta)y_2(\eta) - y_1^{[1]}(\xi)y_2(\xi)$$

The claim now follows by contradiction. Suppose $y_2 > 0$ on (ξ, η), but by our earlier remarks this violates the equality. Similarly, $y_2 < 0$ cannot happen either. Thus $y_2 = 0$ somewhere on (ξ, η). □

The Sturm Separation leads to an interesting classification of solutions to $-(y^{[1]})' + qy = 0$ with singular end points. Namely, if a single nontrivial solution oscillates at an endpoint, all solutions oscillate at the endpoint, otherwise no solution is oscillatory. Therefore we see that oscillatory properties depend on the differential equation, not the solutions!

Definition 1.16 (Oscillatory and Non-Oscillatory)**.** We say that $-(y^{[1]})' + qy = 0$ is oscillatory at a, if for every $x_0 \in J$ there is a nontrivial solution which has infinitely many zeros in the interval (a, c). We say it's non-oscillatory at a if it's not oscillatory a. Swap a for b for the other endpoint definition.

Now that we've seen the zeros are separated, lets talk about comparing equations.

Theorem 1.10 (Sturm Comparison)**.** *Consider*

$$-(p_1y_1')' + q_1y_1 = 0 \quad \& \quad -(p_2y_2')' + q_2y_2 = 0$$

Assume $1/p_i, q_i \in L^1_{loc}(J, \mathbb{R})$ *satisfy* $q_2 \geq q_1, 1/p_2 \geq 1/p_1 > 0$ *on* J. *Suppose that* y_1 *is a non-trivial solution satisfying* $y_1(\xi) = y_1(\eta) = 0$ *for some* $\xi, \eta \in J$ *with* $\xi < \eta$. *Then every solution,* y_2*, of the second equation has a zero in* $[\xi, \eta]$.

Proof. Let $y_1(\xi) = y_1(\eta) = 0$ where $\xi < \eta$ and $y_1 > 0$ on (ξ, η) be a non-trivial solution. Suppose that y_2 is a nontrivial solution of the comparison equation which has no zero on $[\xi, \eta]$. Then a direct computation gives the Picone identity

$$\left[\frac{y_1}{y_2}(p_1y_1'y_2 - p_2y_2'y_1)\right]' = (q_2 - q_1)y_1^2 + (p_1 - p_2)y_1'^2 + p_2\frac{(y_1y_2' - y_2y_1')^2}{y_2^2}$$

Integrating the identity on (ξ, η) yields

$$\int_\xi^\eta (q_2 - q_1)y_1^2 + \int_\xi^\eta (p_1 - p_2)y_1'^2 = -\int_\xi^\eta p_2\frac{(y_1y_2' - y_2y_1')^2}{y_2^2}$$

Notice that the right hand side is positive and the left hand side is negative by our assumption of $q_2 \geq q_1$ and $p_1 \geq p_2 > 0$. This implies $y_1 \equiv 0$, i.e. the trivial solution, thus we've reached a contradiction. Hence, y_2 must have a zero on $[\xi, \eta]$. □

Chapter 2

Characteristics of the SLE

2.1 Oscillation

We'll briefly talk about some of the oscillatory and non-oscillatory properties of the Sturm-Liouville equation in this section. First we begin by defining two possible solutions that occur only for non-oscillatory endpoints

$$(y^{[1]})' + qy = 0$$

with $1/p, q \in L^1_{loc}(J, \mathbb{R}), p > 0$ $a.e.$ on J. Note the sign change due to historical convention.

Definition 2.1. Let u and v be real solutions of the Sturm-Liouville equation. Then

- u is called a principal solution at a if ,

  1. $u(x) \neq 0$ for $x \in (a, \xi)$ and some $\xi \in J$
  2. every solution y which is not a multiple of u satisfies

  $$\frac{u(x)}{y(x)} \to 0, \quad \text{as } x \to a$$

  (note that $y(x) \neq 0$ in some right neighbourhood of a by Sturm Separation)

- v is called a non-principal solution at a if ,

  1. $v(x) \neq 0$ for $x \in (a, \xi)$ and some $\xi \in J$
  2. v is not a principal solution at a.

Notice that if u is a principal solution, then any multiple of u is also a principal solution via the definition. Now we state a few ideas to better classify and normalize solutions.

Lemma 2.2. *The Sturm-Liouville equation is non-oscillatory at a if and only if there exists a principal solution at a.*

Proof. Follows by definition and the Sturm Separation Theorem. □

Theorem 2.1. *Assume the Sturm-Liouville equation is non-oscillatory at a. Let u, v be real solutions satisfying $u(x) \neq 0, v(x) \neq 0$ for all $x \in (a, \xi]$ and some $\xi \in J$. Then*

1. *u is a principal solution at a if and only if*

$$\int_a^\xi \frac{1}{pu^2} = \infty$$

2. *v is a non-principal solution at a if and only if*

$$\int_a^\xi \frac{1}{pv^2} < \infty$$

3. *if u is a principal solution and v is a non-principal solution at a, then there exists $C \in \mathbb{R}$ such that*

$$u(x) = v(x) \int_a^x \frac{C}{pv^2}, \quad x \in (a, \xi]$$

4. *if u is a principal solution and v is a non-principal solution at a, then*

$$|u(t)v(x)| < |u(x)v(t)|, \quad a < t < x \leq \xi$$

Proof. 1 and 2 follow by definition of principal and non-principal solutions. For 3, we'll check by direct computation:

$$(pu')' + qy = 0 \iff \left(pv\frac{C}{pv^2} + pv'\int_a^x \frac{C}{pv^2}\right)' + qv\int_a^x \frac{1}{pv^2} = 0$$
$$\iff \cancel{-\frac{Cv'}{v^2} + \frac{v'C}{v^2}} + ((pv')' + qv)\int_a^x \frac{C}{pv^2} = 0$$

Therefore this is a solution. Since it satisfies the definition of a principal solution, it is one. 4 follows via this formula since the inequality reduces to

$$|u(t)v(x)| < |u(x)v(t)| \iff \int_a^t \frac{1}{pv^2} < \int_a^x \frac{1}{pv^2}, \quad a < t < x \leq \xi$$

□

Remark, in general 2nd order theory we have that if y_1 solves $y'' + py' + qy = 0$. We have that

$$y_2(x) = \int_\xi^x \frac{W[y_1, y_2](s)}{y_1^2(s)} ds$$

Lets get back in touch to reality for minute and consider a classical example

Example 2.3. Consider the two Sturm-Liouville equations

$$y''(x) + \delta y(x) = 0 \quad \& \quad y''(x) + q(x)y(x) = 0$$

on the half-line $J = (a, \infty)$ for some $a \in (1, \infty)$ where $\delta \in \mathbb{R}$. Via the Sturm Comparison Theorem we deduce that

$$q(t) \geq \delta > 0 \implies \text{Oscillatory at } \infty$$

$$q(t) \leq 0 \implies \text{Not-Oscillatory at } \infty$$

since our comparison function is $y(x) = \exp(\pm\sqrt{-\delta}x)$

When both coefficients are present, the best known criteria is the following

Theorem 2.2. *Let* $(y^{[1]})' + qy = 0$ *hold on* J *with* $1/p, q \in L^1_{loc}(J, \mathbb{R})$ *and* $p > 0$ *a.e.. If*

$$\int_a^\xi \frac{1}{p} = \infty \quad \& \quad \int_a^\xi q = \infty$$

for some $\xi \in J$*, then the Sturm-Liouville equation is oscillatory at* a*.*

Proof. Assume that the equation is non-oscillatory at a, and let u, v be positive principal and non-principal solutions on $(a, \eta]$ for some $\eta \in J$. Note that $q = -(v^{[1]})'/v$ on $(a, \eta]$, and integrate by parts:

$$-\int_x^\eta q = \int_x^\eta \frac{(v^{[1]})'}{v} = \frac{v^{[1]}}{v}(\eta) - \frac{v^{[1]}}{v}(x) + \int_x^\eta \frac{1}{p}\frac{(v^{[1]})^2}{v^2} \to -\infty, \quad \text{as } x \to a$$

We deduce

$$\frac{v^{[1]}}{v}(x) \to \infty \quad \text{as } x \to a$$

Thus $v^{[1]}$ is positive near a (since $v > 0$). So v is increasing around a and must have some limit there, denote it by

$$v(x) \to L, \quad \text{as } x \to a, \quad 0 \le L < \infty$$

This means we have some $\xi \in (a, \eta]$, and some $\epsilon > 0$, such that

$$v^2(x) < L^2 + \epsilon, \quad \frac{1}{v^2} > \frac{1}{L^2 + \epsilon}, \quad a < x \le \xi \le \eta$$

But this implies

$$\int_x^\xi \frac{1}{pv^2} \ge \frac{1}{L^2 + \epsilon}\int_x^\xi \frac{1}{p}, \quad a < x \le \xi \le \eta$$

By the hypothesis this means we have

$$\int_a^\xi \frac{1}{pv^2} = \infty$$

which contradicts that v is a non principal solution at a. Thus the equation is oscillatory. □

As previously, there is a version for the right endpoint b. Just swap the sign and integrand domains.

2.2 Floquet Theory

Now you may be wondering what Floquet Theory is? It is simply the study of linear periodic differential systems. As of such, we begin by defining what is know as Hills Equation:

$$-(py')' + qy = \lambda\omega y \quad \text{on } J = \mathbb{R} \text{ or } (a,\infty), \text{ where } a \in \mathbb{R} \tag{2.1}$$

where $1/p, q, \omega \in L^1_{loc}(J, \mathbb{R})$ and $1/p, q, \omega(\cdot + T) = 1/p, q, \omega(\cdot)$. We denote T as the period of the system, i.e. the smallest number such that the coefficients obey the periodic relation. We start of this section with a useful lemma to get to the well known Floquet Theorem.

Lemma 2.4. *For every invertible matrix $A \in GL(n, \mathbb{C})$, there is a matrix $B \in GL(n, \mathbb{C})$ such that $A = \exp(B)$.*

Proof. By conjugation it suffices to consider Jordan matrices, and the Jordan blocks contained in them. Since if D was a diagonal matrix, we have that $\exp(D)$ is just the exponential of the diagonal entries and we're done. Thus we consider $A = \lambda 1 + N$ where N is nilpotent matrix, i.e. $N^m = 0$ for some $m \in \mathbb{N}$, without the loss of generalities. By considering the series expansion for $\ln(1+t)$, and noting that $A = \lambda(1 + N/\lambda)$, we have $A = \exp(B)$ if

$$B = \ln(\lambda)1 + \sum_{n=1}^{m} \frac{(-1)^{n+1}}{n\lambda^n} N^n$$

Note that B is not unique since $B + 2\pi i k 1$ with $k \in \mathbb{Z}$ will also work. □

Definition 2.5. The principal fundamental solution to $\dot{x} = Ax$ is the fundamental solution that satisfies $X(0) = 1$.

Lemma 2.6. *The principal fundamental solution $X(t)$ to $\dot{x} = Ax$ with T-periodic $A(\cdot)$ satisfies*

$$X(nT + t) = X(t)X(T)^n$$

for all $t \in \mathbb{R}$ and all $k \in \mathbb{N}$.

Proof. The claim is trivial for $n = 0$, so we proceed by induction. Suppose it holds for $n - 1 \in \mathbb{N}$. We clearly have

$$X(nT) = X((n-1)T + T) = X(T)X(T)^{n-1} = X(T)^n$$

This leads us to define

$$Y(t) \equiv X(t + nT)X(nT)^{-1}, \quad \forall t \in \mathbb{R}$$

Using this, we'll show that it also holds true for n. We notice that $Y(0) = 1$ just as the principal fundamental solution $X(t)$. Differentiating $Y(t)$ and using it's periodicity reveals

$$\frac{d}{dt}Y(t) = \dot{X}(nT + t)X(nT)^{-1} = A(nT + t)X(nT + t)X(nT)^{-1} = A(t)Y(t)$$

Since the solution of this initial value problem is unique, we deduce that $Y(t) = X(t)$. Therefore, by the definition of $Y(t)$ we have

$$X(nT + t) = X(t)X(nT) = X(t)X(T)^n, \quad \forall t \in \mathbb{R}$$

□

By uniqueness of the solution to the Sturm-Liouville Equation, this lemma showcases that if we reduce Hills equation to the equivalent system that $X(t) = \exp(At)$ where

$$\dot{X} = AX, \qquad A = \begin{pmatrix} 0 & 1/p \\ q - \lambda\omega & 0 \end{pmatrix}$$

hence all the usual nice properties of the exponential function hold.

Theorem 2.3 (Floquet Theorem). *If $X(t)$ is a fundamental matrix solution to $\dot{x} = Ax$, then so is $X(t + T)$. Furthermore, there exists $\Phi(t)$ with T-period such that*

$$\Phi(t) = X(t)\exp(Bt)$$

Proof. By the previous lemma, we know that the first claim follows immediately. We see that we require

$$\Phi(nT+t) = X(nT+t)\exp(B(nT+t)) = X(t)X(T)^n\exp(B(nT))\exp(Bt) = \Phi(t)$$

Thus it suffices to choose B such that

$$X(T)\exp(BT) = 1$$

If we recall lemma (2.4), we can choose such a matrix B since $X(\cdot) \in GL(n,\mathbb{C})$. Thus the claim follows.

□

This theorem gives us that if our data is $L^1(J,\mathbb{C})$ where $|J| = T$ (i.e. the length of J is the length of the period) then Hills equation has a global solution by extending $\Phi(t)$.

Example 2.7 (Mathieu's Equation)**.** Émile Léonard Mathieu [1835-1890]

$$y'' + (a - 2q\cos(2x))y = 0$$

Without causing overlap into boundary value problems, we have to end the section prematurely, but there is a benefit. A key example of study in the area of Sturm-Liouville are periodic and anti-periodic boundary value problems. This falls into a subset of something which we define as coupled boundary value data. Mathematically the condition is written as $y(a) = \pm y(b)$ and $y^{[1]}(a) = \pm y^{[1]}(b)$, where $+$ denotes the periodic case and $-$ denotes the anti-periodic case.

2.3 Boundary Value Problems

Up to this point we've strictly dealt with the "standard" initial data conditions

$$\underbrace{y_1(\xi) = 1 \quad \& \quad y_1'(\xi) = 0}_{Dirichlet} \qquad \text{or} \qquad \underbrace{y_2(\xi) = 0 \quad \& \quad y_2'(\xi) = 1}_{Neumann}$$

Now we'll generalized to the 2-point boundary value problem. The natural way to introduce conditions on the boundary would be to require a condition such as

$$AY(a) + BY(b) = 0 \qquad \text{where} \qquad A, B \in M_{2\times 2}(\mathbb{C}), \quad Y = \begin{pmatrix} y \\ y^{[1]} \end{pmatrix}$$

The boundary conditions dichotomy consists of separated and coupled data. The separated condition is classified exactly as it sounds, a and b have their own separated conditions:

$$A_1 y(a) + A_2 y^{[1]}(a) = 0 \qquad \& \qquad B_1 y(b) + B_2 y^{[1]}(b) = 0$$

This corresponds to

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix} \qquad \& \qquad B = \begin{pmatrix} 0 & 0 \\ B_1 & B_2 \end{pmatrix}$$

We also impose the additional constraint that $\vec{A} = (A_1, A_2) \neq 0$ and $\vec{B} = (B_1, B_2) \neq 0$ (since if $\vec{A}$ or $\vec{B}$ were 0, the solution must be trivial). In recent literature, the common form of separated boundary condition is written as

$$\cos(\alpha) y(a) + \sin(\alpha) y^{[1]}(a) = 0 \qquad \& \qquad \cos(\beta) y(b) + \sin(\beta) y^{[1]}(b) = 0$$

with some $\alpha, \beta \in [0, \pi)$. It's not hard to see these are equivalent formulations up to a multiplicative constant. Loosely speaking, we can find our required gap between cosine and sine then multiply by a constant to achieve any two constants. It may seem like a weird standard, but when working in term of the Prüfer angle, its meaning becomes obvious.

$$\frac{y}{y^{[1]}}(a) = -\tan(\alpha) \quad \& \quad \frac{y}{y^{[1]}}(b) = -\tan(\beta)$$
$$\implies \Theta(x, \cdot) = (\beta - \alpha)\frac{(x-a)}{b-a} + \alpha \mod 2\pi$$

The coupled boundary condition takes the general form of

$$Y(a) = e^{i\gamma} K Y(b) \qquad \text{with} \qquad K \in SL_2(\mathbb{C}) \quad , \quad \gamma \in (-\pi, \pi]$$

where $SL_2(\mathbb{C})$ is the special linear group of 2 by 2 matrices with complex entries. In both cases, we require

$$\text{rank}(A|B) \equiv \text{rank}\begin{pmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \end{pmatrix} = 2$$

We require that the system of boundary conditions are of full rank so we can guarantee that there exists a nonzero solutions to the problem. If the $A|B$ wasn't full rank, there'd exist a reduction to a system that required $y(\cdot) = y^{[1]}(\cdot) = 0$ which can only be satisfied by the trivial solution(i.e. reduction implies $Ay_1 + By_2 = 0$ for $A, B \neq 0$, then $y(\cdot) = y^{[1]}(\cdot) = 0$ implies the single solution is identically zero.)

2.3.1 Regular Right-Definitness

This case corresponds to

$$-(y^{[1]})' + qy = \lambda\omega y \quad \text{on } J$$

with $1/p, q, \omega \in L^1$ and $AY(a) + BY(b) = 0$ where $A, B \in M_{2\times 2}(\mathbb{C})$. This is the most common case and the nicest. The problem for finding the eigenvalues comes down to something called the characteristic function.

Definition 2.8. The Characteristic function δ is defined by

$$\delta(\lambda) = \det[A + B\Phi(b, a, \cdot))], \quad \lambda \in \mathbb{C}$$

Lemma 2.9. *$\delta(\lambda)$ is well defined and entire in λ.*

Proof. Since the primary fundamental matrix Φ exists and is continuous at the endpoints of J this is well defined. We've already shown y and $y^{[1]}$ are entire, hence δ is entire. □

Clearly $\delta(\lambda) = 0$ if and only if λ is an eigenvalue to the Sturm-Liouville Equation here. Since the characteristic function is just a reformulation of the boundary value constraint. Namely

$$AY(a) + BY(b) \to AX(a, \cdot) + BX(b, \cdot) = 0 \iff A + B\Phi(b, a, \cdot)) = 0$$

Thus we need the above not to be invertible, or else we'll achieve the trivial solution. This is where the definition comes from. So let's start with finding the eigenvalues a coupled type problem.

Lemma 2.10. *For the boundary value problem, exactly one of the four cases may occur:*

- *There are no eigenvalues in* $\mathbb{C}$
- *Every complex number is an eigenvalue*
- *There are exactly* n *eigenvalues in* $\mathbb{C}$ *of some* $n \in \mathbb{N}$
- *There are an infinite but countable number of eigenvalues in* $\mathbb{C}$ *and there is no finite accumulation point in* $\mathbb{C}$

Proof. If $\delta(\lambda) = const$ we have 1. If $\delta(\lambda) = 0$ we have 2. If $\delta(\lambda) = (\lambda - \lambda_1)\ldots(\lambda - \lambda_n)$ we have 3. Since δ is entire, we cannot have finite accumulation points. That gives us 4. □

Now we'll introduce a self-adjoint condition to allow us some nice properties in Hilbert space. We call the self-adjoint Sturm-Liouville Problem the boundary value problem that includes the assumption that

$$AEA^* = BEB^* \quad \text{where } E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Theorem 2.4. *Consider the self-adjoint Sturm-Liouville Problem with our* L^1 *data,* p *real-valued, and* $\omega > 0$ *a.e. on* J*. Then all eigenvalues are real.*

Proof. Take any eigenfunction y that satisfies $-(y^{[1]})' + qy = \lambda\omega y$ and consider the weighted inner product.

$$(\lambda - \overline{\lambda}) \int_J |y|^2\omega = \int_J W_p[y, \overline{y}]' = W_p[y, \overline{y}](b) - W_p[y, \overline{y}](a)$$

In both the separated and coupled boundary data cases we have the difference of Wronskians computes to zero. Therefore, since $|y|^2, \omega > 0$, this implies

$$(\lambda - \overline{\lambda}) \int_J |y|^2 \omega = 0 \implies \lambda = \overline{\lambda}$$

Which shows that λ is real. □

A somewhat important remark is that the difference of $W_p[y, \overline{y}](b) - W_p[y, \overline{y}](a)$ is identically zero on any self-adjoint domain. The self-adjoint assumption is therefore crucial for our analysis. We now present that our eigenfunctions correspond to a basis for our Hilbert Space L^2 via the Green's function to the Sturm-Liouville equation.

Theorem 2.5 (Green's Kernel)**.** *Let the Sturm-Liovuille boundary value problem hold, and assume for some $\mu \in \mathbb{R}$ that*

$$\det(A + B\Phi(b, a, \mu)) \neq 0$$

Then we have

- *μ is not an eigenvalue*
- *The Green's function $K(\cdot, \cdot, \mu)$ exists and is hermitian*
- *The integral operator T defined by*

$$(Tf)(x) = \int_J K(x, s, \mu)\omega(s)f(s)ds, \quad f \in L^2(J, \omega), \quad x \in J$$

  is compact (completely continuous) self-adjoint operator defined on L^2 and maps $L^2 \to L^2$

- *λ is an eigenvalue of the Sturm-Liouville Problem if and only if $1/(\lambda - \mu)$ is an eigenvalue of the operator T.*
- *The above eigenvalues have the same eigenfunctions*

- *The operator T and the boundary value problem have a countably infinite number of eigenvalues, they can be indexed to satisfy an ordered inequality.*

Proof. The first few claims are obvious. The relationship between eigenvalues and eigenfunctions follows by solving the inhomogeneous Fredholm integral equation with Liouville-Neumann series. The last bit follows by the spectral theorem for compact self-adjoint operators. □

Note that $1/(\lambda - \mu)$ generally takes the name of the resolvent. It is used to study the spectral properties of operators on Banach spaces. It may be used to solve Fredholm integral equation such as T as above.

Now let's find the behaviour of zeros and eigenvalues of a separated type problem. We'll do this by means of the Prüfer transformation since it's cleaner.

Lemma 2.11. *Let $p > 0$ a.e. on J and assume $\omega \geq 0$ a.e. $1/p, q, \omega \in L^1(J, \mathbb{R})$. The Sturm-Liouville Equation has zero's like*

$$N(\lambda) \sim \frac{\sqrt{\lambda}}{\pi} \int_J \sqrt{\frac{\omega}{p}},$$

Proof. The proof is somewhat technical and relies on approximating our L^1 functions with nice C^1 functions. Let $h \in C^1(J)$ s.t. h By earlier remarks, we can assume $p \equiv 1$ without the loss of generality. The asymptotics here don't depend depend on our choice of solution so we choose the Dirichlet solution, $y(a, \lambda, \cdot) = 1$ and $y'(a, \lambda, \cdot) = 0$ for convenience. We define a Prüfer angle $\Theta = \Theta(x, \lambda, \cdot)$ by

$$\tan \Theta = -\frac{y'}{yh\sqrt{\lambda}}, \quad \lambda > 0$$

This gives us $\Theta(a, \lambda, \cdot) = 0$ and

$$\Theta' = h\sqrt{\lambda} + \sqrt{\lambda}\left(\frac{\omega}{h} - h\right)\cos^2\Theta - \frac{h'}{2h}\sin 2\Theta - \frac{q}{h\sqrt{\lambda}}\cos^2\Theta$$

We have that Θ increases at zeros of y, thus

$$N(\lambda) = \frac{\Theta(b, \lambda)}{\pi} + \mathcal{O}(1), \quad \lambda > 0$$

Since $N(\lambda)$ is independent of h, if we can choose h such that for any $\epsilon > 0$, we have

$$\int_J |h - \sqrt{\omega}| < \epsilon, \quad \& \quad \int_J \left|\frac{\omega}{h} - h\right| < \epsilon$$

So lets show how to choose such an h. First choose a function g such that

$$\int_J |\sqrt{\omega} - g|^2 < |J|\delta^2, \quad \omega \geq 0$$

i.e. $||\sqrt{\omega} - g||_\infty < \delta$ (it is possible to find a polynomial with this property). After fixing such a g, define

$$h \equiv g + \delta$$

so that $g > 0$ and $g \in C^1$. This implies that we have

$$\int_J |h - \sqrt{\omega}| \leq 2|J|\delta$$

Note that we can bound the second term in the Prüfer equation by

$$\left|\frac{\omega}{h} - h\right| \leq \frac{\sqrt{\omega}}{h}|h - \sqrt{\omega}| + |h - \sqrt{\omega}|$$

Thus we now only have to deal with the first term in the above inequality. To tackle this, we create the following two sets:

$$I_{\geq} = \{x \in J : g \geq \sqrt{\delta}\}$$

$$I_{<} = \{x \in J : g < \sqrt{\delta}\}$$

For $I_{\geq}$, we have $h > \sqrt{\delta}$. If we invoke Cauchy-Schwarz on the integral, we see

$$\begin{aligned}\int_{I_{\geq}} \frac{\sqrt{\omega}}{h}|h - \sqrt{\omega}| &\leq \left(\int_{I_{\geq}} \frac{\omega}{h^2}\right)^{1/2} \left(\int_{I_{\geq}} |h - \sqrt{\omega}|^2\right)^{1/2} \\ &\leq \left(\frac{||\omega||_1}{\delta}\right)^{1/2} (4|J|\delta^2)^{1/2} \\ &= 2\sqrt{||\omega||_1 \delta |J|}\end{aligned}$$

This allows us to control the first region, for the second we have $h \geq \delta$. Thus applying Cauchy-Schwarz again gives

$$\int_{I_<} \frac{\sqrt{\omega}}{h}|h - \sqrt{\omega}| \leq \frac{1}{\delta}\left(\int_{I_<} \omega\right)^{1/2} \delta\sqrt{|J|}$$

By the Minkowski inequality and the fact $g < \sqrt{\delta}$ we have that

$$\int_{I_<} \omega \leq 2\int_{I_<} (\sqrt{\omega} - g)^2 + 2\int_{I_<} g^2 \leq 2|J|\delta^2 + 2|J|\delta$$

Now if we put it all together, we have that

$$\int_J \frac{\sqrt{\omega}}{h}|h - \sqrt{\omega}| \leq 2\delta|J| + \underbrace{2\sqrt{||\omega||_1\delta|J|}}_{I_\geq} + \underbrace{\sqrt{|J|}(2|J|\delta^2 + 2|J|\delta)}_{I_<}$$

This allows us to control everything. Thus, given $\epsilon > 0$, we choose $\delta > 0$ such that

$$2\delta|J| + 2\sqrt{||\omega||_1\delta|J|} + \sqrt{|J|}(2|J|\delta^2 + 2|J|\delta) < \frac{\epsilon}{2}$$

The following choice reduces Prüfer's equation to

$$\left|\int_J \sqrt{\omega} - \frac{\Theta(b,\lambda)}{\sqrt{\lambda}}\right| < \epsilon + \frac{1}{\sqrt{\lambda}}\int_J \left|\frac{h'}{2h}\right| + \frac{1}{\lambda}\int_J \frac{|q|}{h}$$

Thus, as λ becomes larger the latter terms are negligible. Therefore the right hand side is bounded by any ϵ which shows

$$\Theta(b,\lambda) \sim \sqrt{\lambda}\int_J \sqrt{\omega} \implies N(\lambda) \sim \frac{\sqrt{\lambda}}{\pi}\int_J \sqrt{\omega}$$

□

Corollary 2.1. *The eigenvalues of the regular separated boundary value problem have asymptotics like*

$$\lambda \sim n^2\pi^2\left(\int_J \sqrt{\frac{\omega}{p}}\right)^{-2}$$

With a little bit more work, we can extend to the case where p changes sign. Define

$$p^+(t) = \frac{|p(t)| + p(t)}{2} \quad \& \quad p^-(t) = \frac{|p(t)| - p(t)}{2}$$

i.e. the positive and negative parts of p respectively with 0 everywhere else.

Lemma 2.12. *If p changes sign, then we have that*

$$\underbrace{\lambda_n \sim n^2\pi^2 \left(\int_a^b \sqrt{\frac{\omega}{p^+}}\right)^{-2}}_{n\to\infty} \quad \& \quad \underbrace{\lambda_n \sim -n^2\pi^2 \left(\int_J \sqrt{\frac{\omega}{p^-}}\right)^{-2}}_{n\to-\infty}$$

Proof. It follows almost verbatim from the previous proof, simply split the interval J where the Sturm-Liouville equation is posed i.e. $p^+ - p^- = p$. □

Now that we've covered some of the specifics, we have the following

Corollary 2.2. *If $p > 0$ on J, we have that the eigenvalues of the problem can be numbered and ordered as*

$$-\infty < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \to \infty$$

if the boundary conditions are coupled. For the separated case we have strict inequality. If p changes sign we have

$$-\infty \leftarrow \ldots \leq \lambda_{-2} \leq \lambda_{-1} \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \to \infty$$

if the boundary conditions are coupled. For the separated case we have strict inequality.

The eigenfunctions corresponding to λ_n in the first case have reasoning in the indexing. By the Sturm comparison theorem we have that n roughly corresponds to $n \pm 1$ or n zeros on J. Note that we haven't proved the existence of such eigenvalues here, but the reader may find the proofs readily available online.

2.3.2 Regular Left-Definiteness

Notice that we required $\omega \geq 0$ so far. To brush some of left definite problem under the rug and deal with it another day, we remark that we can deal with the left-definite case as a right-definite one by dealing with a 2-parameter spectral problem. More specially, consider

$$-(y^{[1]})' + \underbrace{(q - \lambda\omega)}_{\tilde{q}}\, y = \xi|\omega|y$$

where ξ is our second spectral parameter. Clearly if $\omega \in L^1(J)$ we have that $\tilde{q} \in L^1(J)$ as well. So everything we just covered applies to ξ. This is the beginning of basic eigen-curve methods to find the spectrum of our left-definite problem. I.e, since the ODE depends continuously on the spectral parameter, we'll be able to find a curve $\xi(\lambda)$ with the property that $\xi(\lambda_n) = 0$ implies λ_n is an eigenvalue of the problem. The only problem with this is that you can show there are two roots for $\xi(\lambda_n) = 0$, so to insure the problem is well posed we restrict that $\xi_0(|\omega|) > 0$ when $p > 0$ to restrict to a connected component. Thus, via our ordering with the regular problem we have restrict to the connected component for all $\xi_n(|\omega|)$ As you probably realize by this point, the conditions on the coefficient data strongly impact the techniques of analysis we can utilize. Even the previous definition isn't very enlightening, so lets see what else we have to work with. Let D_{max} and $D(A, B)$ be the linear sub manifolds of the Hilbert space $\mathcal{H} \equiv L^2(J, |\omega|)$ defined by

$$D_{max} = \left\{y \in \mathcal{H} : y, y^{[1]} \in AC_{loc}(J), \mathcal{L}(y) = \frac{1}{|\omega|}(-(y^{[1]})' + qy) \in \mathcal{H}\right\}$$

$$D(A, B) = \left\{y \in D_{max} : AY(a) + BY(b) = 0\right\}$$

i.e. D_{max} is a domain of solutions to the Sturm-Liouville Problem and $D(A, B)$ is our domain of boundary value solutions. Note that the end points are defined by limiting behaviour near them since they may be singular. We have our indefinite inner product on $\mathcal{H}$ via

$$(f, g) = \int_J f(x)g(x)\omega(x)dx$$

If we consider the Sturm-Liouville Equation, we see

$$\lambda(y, y) = \int_J (-(y^{[1]})' + qy)\overline{y} = \int_J \underbrace{\frac{(-(y^{[1]})' + qy)}{\sqrt{\omega}}}\underbrace{\overline{y}\sqrt{\omega}} \leq ||\mathcal{L}(y)||||y||, \quad y \in D(A, B)$$

Thus we have that $\lambda(y, y)$ is a well defined functional on $D(A, B)$. These two functionals give us the key characterization between these cases. In the right-definite case, we have that $(y, y) > 0$ for all $y \in D(A, B)$ excluding $y \equiv 0$ (or $(y, y) < 0$ for all $y \in D(A, B)$, we retrieve the other case with a simple sign change). For the left-definite case, we have the following

Theorem 2.6. *The Sturm-Liouville Problem is left-definite if and only if* $\lambda(y, y) > 0$ *for all* $y \in D(A, B)$ *excluding zero.*

Proof. Consider the the 2 parameter regularization,i.e. $-(y^{[1]})' + (q - \lambda\omega)y = \xi|\omega|y$. We clearly have

$$\xi_0(|\omega|) = \inf \frac{\lambda(y, y)}{\int_J |y|^2|\omega|}$$

Therefore the statement follows. □

What can we say about the eigenvalues and eigenfunctions of these left-definite problems? Basically everything we had for the right-definite case. If we restrict $p > 0$, we have our nice change of variables and many of the previous arguments hold. When p changed sign, we could split the Sturm-Liouville equation, the exact same argument holds for ω changing sign thus.

Corollary 2.3. *Consider the left-definite Sturm-Liouville problem. Let* $p > 0$ *on* J, *we have that the eigenvalues of the problem can be numbered and ordered as*

$$-\infty \leftarrow \ldots \leq \lambda_{-2} \leq \lambda_{-1} \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \to \infty$$

if the boundary conditions are coupled. For the separated case we have strict inequality. The eigenvalues satisfy the following asymptotics

$$\lambda_{\pm n} \sim \pm n^2\pi^2 \left(\int_J \sqrt{\frac{\omega^{\pm}}{p}} \right)^{-2}$$

where $\omega^{\pm} = (|\omega| \pm \omega)/2$.

The proof is very similar to the right-definite case, and is omitted. We're obviously missing some of the key details of this argument, but it's the rough idea.

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