Assignment 6

MATC34 – Complex Variables – Fall 2015

SOLUTIONS

Question 1 Using Cauchy inequalities for derivatives of analytic functions, verify the following statement. Let $f(z)$ be an entire function. Assume that $|f(z)| \leqslant |z| + 1$ for any z. Then function $f(z)$ is linear, i.e. $f(z) = az + b$ where a and b are constants.

Solution From Cauchy's Integral formula we know

$$\left|f^{(n)}(z_0)\right| = \left|\frac{n!}{2\pi i}\int_{C_R}\frac{f(z)}{(z-z_0)^{n+1}}dz\right| \leqslant \frac{n!}{2\pi}\int_{C_R}\frac{|z|+1}{|z-z_0|^{n+1}}dz \leqslant n!\frac{R+|z_0|+1}{R^n} \quad \forall R > 0$$

where C_R is the circle of radius R entered at z_0. Thus if we take any $n \geqslant 2$, we have that

$$\left|f^{(n)}(z_0)\right| \leqslant \lim_{R\to\infty} n!\frac{R+|z_0|+1}{R^n} = 0 \quad \forall z_0 \in \mathbb{C}$$

Basically, we see that $f''(z) = 0$ for all z. Integrate the equation twice and we see

$$f(z) = az + b, \quad a, b \in \mathbb{C}$$

□

Question 2 Obtain the Taylor series

$$f(z) = e^z = e\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!}$$

a)using the derivatives $f^{(n)}(1)$, $n = 0, 1, 2, 3, \ldots$ b) using the identity $e^z = ee^{z-1}$

Solution a) We know the taylor series of $f(z) = e^z$ is given by

$$f(z) = \sum_{n=0}^{\infty}\frac{f^{(n)}(1)}{n!}(z-1)^n$$

Since we know $\frac{d}{dz}e^z = e^z$, we see that $f^{(n)}(1) = e$ for all n. Thus

$$f(z) = \sum_{n=0}^{\infty}\frac{f^{(n)}(1)}{n!}(z-1)^n = e\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!}$$

b) We know the taylor series of e^z is given by

$$e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!}$$

Thus

$$f(z) = ee^{z-1} = e\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!}$$

□