Assignment 5

MATC34 – Complex Variables – Fall 2015

SOLUTIONS

Question 1 Evaluate the integral using Cauchy's formula:

$$\int_C \frac{\tan(z/2)}{(z-x_0)^3} dz$$

where $x_0 \in \mathbb{R} \setminus \{\pm 2\}$, and C is the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$, $y = \pm 2$.

Solution Recall Cauchy's Integral Formula: Let $z_0 \in D$ and $f(z)$ be holomorphic in D, then if $\partial D = C$, we have

$$\int_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

Thus, in our case we see $f(z) = \tan(z/2)$ is holomorphic inside $D = ||z||_\infty < 2$. If we have $|x_0| > 2$, then

$$\frac{\tan(z/2)}{(z-x_0)^3} \quad \text{is holomorphic in } D \implies \int_C \frac{\tan(z/2)}{(z-x_0)^3} dz = 0$$

If $|x_0| < 2$, the integral is just as in Cauchy's formula with $n = 2$, thus

$$\int_C \frac{\tan(z/2)}{(z-z_0)^3} dz = \frac{2\pi i}{2!} \frac{d^2}{dz^2} \tan\left(\frac{z}{2}\right)\Big|_{z=z_0} = \pi i \frac{d}{dz}\left(\frac{1}{2}\sec^2\left(\frac{z}{2}\right)\right)\Big|_{z=x_0} = \frac{\pi i}{2} \tan\left(\frac{x_0}{2}\right) \sec^2\left(\frac{x_0}{2}\right)$$

Thus

$$\boxed{\int_C \frac{\tan(z/2)}{(z-x_0)^3} dz = \begin{cases} \frac{\pi i}{2} \tan\left(\frac{x_0}{2}\right) \sec^2\left(\frac{x_0}{2}\right) & |x_0| < 2 \\ 0 & |x_0| > 2 \end{cases}}$$

□

Question 2 Evaluate the integral

$$\int_{C_i} \frac{dz}{(z-1)^2(z-i)}$$

Where a) $C_1 = \{z : |z-1| = 1\}$, b) $C_2 = \{z : |z - i/2| = 1\}$ and c) $C_3 = \{z : ||z||_\infty = 2\}$.

Solution For a), it is clear that $1 \in D_1 = \{z : |z-1| < 1\}$ and $\partial D_1 = C_1$, thus Cauchy's Formula applies with $f(z) = 1/(z-i)$ and $n = 1$. Thus

$$\int_{C_1} \frac{dz}{(z-1)^2(z-i)} = \frac{2\pi i}{1!} \frac{d}{dz} \frac{1}{z-i}\Big|_{z=1} = -\frac{2\pi i}{(1-i)^2} = \pi$$

For b), it is clear that $i \in D_2 = \{z : |z - i/2| < 1\}$ and $\partial D_2 = C_2$, thus Cauchy's Formula applies with $f(z) = 1/(z-1)^2$ and $n = 0$. Thus

$$\int_{C_2} \frac{dz}{(z-1)^2(z-i)} = \frac{2\pi i}{0!} \frac{1}{(z-1)^2}\Big|_{z=i} = \frac{2\pi i}{(i-1)^2} = -\pi$$

For c), notice that $1, i \in D_3 = \{z : ||z||_\infty < 2\}$ and $\partial D_3 = C_3$. Since $\frac{1}{(z-1)^2(z-i)}$ is holomorphic on $D_3 \setminus \{i, 1\}$, we may decompose the integral into two pieces, specifically:

$$\int_{C_3} \frac{dz}{(z-1)^2(z-i)} = \int_{C_1} \frac{dz}{(z-1)^2(z-i)} + \int_{C_2} \frac{dz}{(z-1)^2(z-i)} = \pi - \pi = 0$$

□

Q3 Evaluate the integral using Cauchy's formula:

$$\int_{|z-2i|=1} \frac{z+i}{z^3+2z^2} dz$$

Solution We see the function inside the integral takes the form

$$f(z) = \frac{z+i}{z^2(z+2)}$$

which is holomorphic on $\mathbb{C} \setminus \{0, -2\}$. Notice that neither 0 or -2 live in $D = \{z : |z-2i| < 1\}$, thus

$$\boxed{\int_{|z-2i|=1} \frac{z+i}{z^3+2z^2} dz = 0}$$

by Cauchy's integral formula. □