Tutorial 8

MAT334 – Complex Variables – Spring 2016
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SOLUTIONS

2.5 - # 1,5 Locate each of the singularities of the given functions and tell whether it is a removable singularity, a pole, or an essential singularity. If the singularity is removable, give the value of the function at the point;if the singularity is a pole, give the order of the pole.

$$\frac{e^z - 1}{z} \quad \& \quad \frac{2z+1}{z+2}$$

Solution We know that

$$e^z = \sum_{n \geqslant 0} \frac{z^n}{n!}$$

Thus

$$\frac{e^z - 1}{z} = \sum_{n \geqslant 1} \frac{z^{n-1}}{n!} = 1 + \frac{z}{2} + \mathcal{O}(z^2)$$

So the singularity of the function is at $z = 0$ and removable, the value of the function at $z = 0$ is 1. For the second function, note

$$\frac{2z+1}{z+2} = \frac{-3}{z+2} + 2$$

is the Laurent series for the function. Thus we see the simple pole at $z_0 = -2$. □

2.5 - # 7,10 Find the Laurent series for the given function about the indicated pint. Also, give the residue of the function at the point,

$$\frac{e^z - 1}{z^2}; \quad z_0 = 0 \quad \& \quad \frac{z}{(\sin z)^2}; \quad z_0 = 0 (\text{four terms of the Laurent series})$$

Solution From the previous question, we easily see that

$$\frac{e^z - 1}{z^2} = \sum_{n \geqslant 1} \frac{z^{n-2}}{n!} = \frac{1}{z} + \frac{1}{2} + \mathcal{O}(z^2)$$

We read the residue off the $1/z$ term as $\text{Res}(f; 0) = 1$. For the second function, we compute $z/\sin z = a_0 + a_2 z^2 + a_4 z^4 + \mathcal{O}(z^6)$.

$$\begin{aligned} z = \sin z \frac{z}{\sin z} &= \left(z - \frac{z^3}{6} + \frac{z^5}{120} - \mathcal{O}(z^7) \right) \left(a_0 + a_2 z^2 + a_4 z^4 + \mathcal{O}(z^6) \right) \\ &= a_0 z + (a_2 - 1/6) z^3 + (a_4 - a_2/6 + 1/120) z^5 + \mathcal{O}(z^7) \end{aligned}$$

Thus

$$\frac{z}{\sin z} = 1 + \frac{z^2}{6} + \frac{7}{360} z^4 + \mathcal{O}(z^6)$$

So we see

$$\frac{z}{(\sin z)^2} = \left(\frac{z}{\sin z}\right)\left(\frac{1}{\sin z}\right) = \left(1 + \frac{z^2}{6} + \frac{7}{360}z^4 + \mathcal{O}(z^6)\right)\left(\frac{1}{z} + \frac{z}{6} + \frac{7}{360}z^3 + \mathcal{O}(z^5)\right)$$
$$= \frac{1}{z} + \frac{z}{3} + \frac{z^3}{15} + \mathcal{O}(z^5)$$

□

2.5 - # 15 If f is analytic in $0 < |z - z_0| < R$ and has a pole or order k at z_0, show that

$$\text{Res}\left(\frac{f'}{f}; z_0\right) = -k$$

Solution Write

$$f(z) = \frac{g(z)}{(z - z_0)^k}$$

where $g(z)$ is analytic in $|z - z_0| < R$ and $g(z_0) \neq 0$. We see that

$$\frac{f'}{f} = \frac{g'}{g} - \frac{k}{(z - z_0)}$$

Since g is analytic, we know that g'/g is analytic around z_0. Thus we simply read the coefficient off the $1/(z-z_0)$ to find the residue.

$$\text{Res}\left(\frac{f'}{f}; z_0\right) = \text{Res}\left(-\frac{k}{(z - z_0)} + \frac{g'}{g}; z_0\right) = -k$$

□

2.5 - # 18 Here is an alternate proof that

$$\frac{1}{2\pi i}\int_{|z-z_0|=r} f(z)dz = \text{Res}(f; z_0)$$

is independent of r. Assume that $z_0 = 0$ with no loss of generality. We have

$$\frac{\partial}{\partial t}(f(re^{it})e^{it}) = ie^{it}f(re^{it}) + ire^{2it}f'(re^{it})$$

and that

$$\frac{d}{dr}\left[\frac{1}{2\pi}\int_0^{2\pi} rf(re^{it})e^{it}dt\right] = \frac{1}{2\pi}\int_0^{2\pi}[f(re^{it}) + rf'(re^{it})e^{it}]e^{it}dt$$

Thus

$$\frac{d}{dr}\left[\frac{1}{2\pi}\int_0^{2\pi} rf(re^{it})e^{it}dt\right] = \frac{1}{2\pi i}\int_0^{2\pi}\frac{\partial}{\partial t}(f(re^{it})e^{it})dt = \frac{f(r) - f(r)}{2\pi i} = 0$$

We then can conclude that

$$\frac{1}{2\pi i}\int_{|z-z_0|=r} f(z)dz = \frac{1}{2\pi}\int_0^{2\pi} f(re^{it})re^{it}dt$$

is independent of r. □

2.5 - # 22 b),d) Find the Laurent series about $z_0 = 0$ for the following function, valid in the indicated regions.

$$z^4 \sin\left(\frac{1}{z}\right); \quad 0 < |z| < \infty \quad \& \quad \exp\left(z + \frac{1}{z}\right); \quad 0 < |z| < \infty$$

Solution Recall

$$\sin z = \sum_{n\geqslant 0} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

Thus

$$z^4 \sin\left(\frac{1}{z}\right) = \sum_{n\geqslant 0} \frac{(-1)^n z^{3-2n}}{(2n+1)!}$$

For the second function, we note that

$$\exp\left(z+\frac{1}{z}\right) = \exp(z)\exp\left(\frac{1}{z}\right) = \left(\sum_{n\geqslant 0}\frac{z^n}{n!}\right)\left(\sum_{m\geqslant 0}\frac{1}{m!z^m}\right)$$

multiplying out the series product gives

$$\exp\left(z+\frac{1}{z}\right) = \sum_{k\in\mathbb{Z}} c_k z^k \quad \text{where } c_k = \sum_{n-m=k}\frac{1}{n!m!} = \sum_{m\geqslant\max(0,-k)}\frac{1}{(m+k)!m!}$$

or we may use the binomial theorem to see

$$\exp(z+1/z) = \sum_{k\geqslant 0}\frac{(z+1/z)^k}{k!} = \sum_{k\geqslant 0}\frac{1}{k!}\sum_{n=0}^{k}\binom{k}{n} z^{k-2n} = \sum_{k\geqslant 0}\sum_{n=0}^{k}\frac{z^{k-2n}}{(k-n)!n!}$$

□

2.5 - # 27 a),c) Classify the nature of the singularity at ∞ of each of the following functions. If the singularity is removable, give the value at ∞. If the singularity is a pole, give its order; in each case find the first few terms in the Laurent series about ∞.

$$3z^2 + 4 - \frac{1}{z} \quad \& \quad \frac{z^2}{z-4}$$

Solution To consider ∞, we introduce the change of variables $z = 1/w$. Then the functions look like

$$\frac{3}{w^2} + 4 - w \quad \& \quad \frac{1}{w(1-4w)} = \sum_{n\geqslant 0}(4w)^{n-1}$$

Thus we see that the first function has a pole of order 2 at ∞ and the second function has a simple pole at ∞. □