Tutorial 7

MAT334 – Complex Variables – Spring 2016

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Solutions

2.3 - # 3 Evaluate

$$\int_{|z+1|=2} \frac{z^2}{4-z^2} dz$$

Solution Note we may rewrite the integrand as

$$\int_{|z+1|=2} \frac{z^2}{4-z^2} dz = \int_{|z+1|=2} \underbrace{\frac{z^2}{2-z}}_{=f(z)} \frac{dz}{z+2}$$

and notice that $-2 \in \{z : |z+1| < 2\}$. Thus Cauchy's Integral Formula tells us

$$\int_{|z+1|=2} \frac{z^2}{4-z^2} dz = 2\pi i f(-2) = 2\pi i$$

$\square$

2.3 - # 7 Evaluate

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta}, \quad a > b > 0$$

Solution Define $\gamma = \{z : |z| = b\}$, then

$$b\cos\theta = \frac{1}{2}\left(z + \frac{b^2}{z}\right) = \frac{z^2+b^2}{2z} \quad \text{since } z = be^{i\theta} \text{ on } \gamma$$

Therefore, by definition of parameterizing a path integral, we have (note $dz = izd\theta$).

$$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{1}{i}\int_\gamma \frac{2dz}{2az + z^2 + b^2}$$

We may factor the polynomial in the bottom via the quadratic formula, we see

$$z^2 + 2az + b^2 = 0 \implies z_\pm = -a \pm \sqrt{a^2 - b^2}$$

Note since $a > b > 0$, we have that

$$-a + \sqrt{a^2-b^2} \in \{z : |z| < b\}$$

Thus Cauchy's Integral Formula gives us

$$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

$\square$

2.3 - # 12 Evaluate

$$\int_\gamma \sin z dz$$

where γ is any curve joining i to π.

Solution Note that

$$\frac{d}{dz}(-\cos z) = \sin z dz$$

Thus

$$\int_\gamma \sin z dz = -\cos z\Big|_i^\pi = -\cos \pi + \cos i = 1 + \cosh(1)$$

□

2.3 - # 16 Prove if f is analytic and never zero on a domain D, then $|f(z)|$ has no local minima in D. That is, the graph $(x, y, |f(x+iy)|)$ has no "pits". Use the fact that an analytic function g cannot have a strict local maximum by

$$|g(z)| \leqslant \max_{\theta\in[0,2\pi)} |g(z+re^{i\theta})|$$

where r is small enough so $z + re^{i\theta}$ is within the domain of analyticity.

Solution Since f is non-zero on D, consider

$$g = \frac{1}{f}$$

Note g is analytic since f is analytic. A local maxima of g corresponds to a local minima of f. Using the fact stated the in question allows us to deduce that g has cannot have strict local maximum, which implies that f cannot have a local minima in D. □

2.4 - # 6,8 Find the order of each zero for the given functions

$$\mathrm{Log}(1-z), \quad |z| < 1 \quad \& \quad \frac{z}{z^2+1}$$

Solution Using a geometric series, we see

$$-\mathrm{Log}(1-z) = \int \frac{dz}{1-z} = \int \sum_{n\geqslant 0} z^n dz = \sum_{n\geqslant 0} \int z^n dz = z\sum_{n\geqslant 0} \frac{z^n}{n+1}$$

Thus $\mathrm{Log}(1-z)$ has a simple pole at $z = 0$(note the use of Fubini's to exchange the integral and sum). The second function has simple zero at $z = 0$, and simple poles at $z = \pm i$. □

2.4 - # 16 Find the first four terms of the power series expansion around $z_0 = 0$ for $\tan z$

Solution We know that

$$\tan z = \frac{\sin z}{\cos z} \quad \& \quad \cos z = 1 - \frac{z^2}{2} + \frac{z^4}{24} - \mathcal{O}(z^6) \quad \& \quad \sin z = z - \frac{z^3}{6} + \frac{z^5}{5!} - \mathcal{O}(z^7)$$

Thus we'll simply find the power series for $1/\cos z$ and then multiply the series together. To find $\sum a_n z^n = 1/\cos z$, we know that

$$1 = \cos z \frac{1}{\cos z} = \left(1 - \frac{z^2}{2} + \frac{z^4}{24} - \mathcal{O}(z^6)\right)\left(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \mathcal{O}(z^5)\right)$$

expanding out to fourth order, shows

$$1 = a_0 + a_1 z + \left(a_2 - \frac{a_0}{2}\right) z^2 + \left(-\frac{a_1}{2} + a_3\right) z^3 + \left(\frac{a_0}{24} - \frac{a_2}{2} + a_4\right) z^4 + \mathcal{O}(z^5)$$

Thus we see that $a_0 = 1$, $a_1 = 0$, $a_2 = 1/2$, $a_3 = 0$, $a_4 = 5/24$, so

$$\frac{1}{\cos z} = 1 + \frac{z^2}{2} + \frac{5}{24} z^4 + \mathcal{O}(z^5)$$

Thus

$$\begin{aligned} \tan z =& \left(1 + \frac{z^2}{2} + \frac{5}{24} z^4 + \mathcal{O}(z^5)\right)\left(z - \frac{z^3}{6} + \frac{z^5}{120} - \mathcal{O}(z^7)\right) \\ =& z + \frac{z^3}{3} + \frac{2}{15} z^5 + \mathcal{O}(z^7) \end{aligned}$$

□

2.4 - #21 Suppose that f is an entire function and that there are positive constants A, m with

$$|f(z)| \leqslant A|z|^m \quad \text{if } |z| \geqslant R_0$$

Show that f is a polynomial of degree m or less.

Solution By Cauchy's Integral formula, we know that

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

where γ is a closed curve containing z. For any $z \in \mathbb{C}$, define $\zeta = z + Re^{i\theta}$ with $R \geqslant R_0 + |z|$, so we have that

$$|f^{(n)}(z)| \leqslant \frac{n!}{2\pi} \int_\gamma \left| \frac{f(\zeta)}{(\zeta - z)^{n+1}} \right| |d\zeta| \leqslant \frac{An!}{2\pi} \int_0^{2\pi} \frac{|z + Re^{i\theta}|^m}{R^n} d\theta \leqslant Const \cdot R^{m-n}$$

If we take $n > m$, we see in limit as $R \to \infty$ that

$$f^{(n)}(z) = 0 \quad \forall z \in \mathbb{C}$$

Take $z_0 \in \mathbb{C}$, then since f is entire, we have that

$$f^{(k)}(z) = f^{(k)}(z_0) + \int_{z_0}^{z} f^{(k+1)}(\zeta) d\zeta$$

for $k \in \mathbb{N}$, thus

$$f(z) = a_m z^m + \ldots a_1 z + a_0, \quad a_k \in \mathbb{C}$$

□