Tutorial 6

MAT334 – Complex Variables – Spring 2016

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Solutions

2.1 - # 21 Let γ be a piecewise smooth simple closed curve, and suppose that F is analytic on some domain containing γ. Show that

$$\int_\gamma F(z)dz = 0$$

Solution By Green's theorem, we find that (letting $F = u + iv$ at the third step)

$$\int_\gamma F(z)dz = \int_\gamma F(x,y)dx + iF(x,y)dy = \int_D \left[i\frac{\partial F}{\partial x} - \frac{\partial F}{\partial y} \right] dxdy = \int_D \left[\underbrace{i(u_x - v_y) - (v_x + u_y)}_{=0,\text{by CR}} \right] dxdy = 0$$

2.2 - # 2,6 Find the radius of convergence of the given power series

$$\sum_{k=0}^\infty \frac{(k!)^2}{(2k)!}(z-2)^k, \quad \sum_{k=1}^\infty \frac{(2k)(2k-2)\dots 4\cdot 2}{(2k-1)(2k-3)\dots 3\cdot 1} z^k$$

Solution By application of the ratio test to both series, we see on the first

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \frac{((k+1)!)^2/(2k+2)!}{(k!)^2/(2k)!}|z-2| = \lim_{k\to\infty} \frac{(k+1)^2}{(2k+2)(2k+1)}|z-2| = \lim_{k\to\infty} \frac{(1+1/k)^2}{(2+2/k)(2+1/k)}|z-2| = \frac{|z-2|}{4}$$

Thus we see the first sum needs $|z-2| < 4$ to converge. Note for the second we have

$$a_{k+1} = \frac{(2k+2)(2k)(2k-2)\dots 4\cdot 2}{(2k+1)(2k-1)(2k-3)\dots 3\cdot 1} z^{k+1} = \frac{2k+2}{2k+1} z a_k$$

Thus

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \frac{2k+2}{2k+1}|z| = |z|$$

So we see the second sum needs $|z| < 1$ to converge.

2.1 - # 8,10 Find the power series about the origin for the given function.

$$z^2 \cos z, \quad \& \quad \frac{1+z}{1-z} \quad \text{when } |z| < 1$$

Solution We already know the power series of $\cos z$ (by exponential) and $1/(1-z)$ (by geometric series)

$$\cos z = \sum_{n=0}^\infty (-1)^n \frac{z^{2n}}{(2n)!} \quad \& \quad \frac{1}{1-z} = \sum_{n=0}^\infty z^n \quad \text{when } |z| < 1$$

Thus

$$z^2 \cos z = \sum_{n=0}^\infty (-1)^n \frac{z^{2n+2}}{(2n)!} \quad \& \quad \frac{1+z}{1-z} = \sum_{n=0}^\infty (1+z) z^n = 1 + 2\sum_{n\geqslant 1} z^n$$

2.1 - # 17 Find a closed form (that is, a simple expression) for each of the given power series

$$\sum_{n=0}^{\infty}(-1)^n\frac{(z-2\pi i)^n}{n!}, \quad \& \quad \sum_{n=2}^{\infty} n(n-1)z^n \quad \text{(hint:divide by } z^2\text{)}$$

Solution Recall that

$$e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!} \implies e^{-z} = \sum_{n=0}^{\infty}(-1)^n\frac{z^n}{n!} \implies e^{-z} = e^{-z+2\pi i} = \sum_{n=0}^{\infty}(-1)^n\frac{(z-2\pi i)^n}{n!}$$

For the second, we see

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \implies \frac{d^2}{dz^2}\left[\frac{1}{1-z}\right] = \sum_{n=2}^{\infty} n(n-1)z^{n-2} = \frac{1}{z^2}\sum_{n=2}^{\infty} n(n-1)z^n$$

Thus

$$\sum_{n=2}^{\infty} n(n-1)z^n = z^2\frac{d^2}{dz^2}\left[\frac{1}{1-z}\right] = \frac{2z^2}{(1-z)^3}$$

Series Products and Quotients Consider

$$f = \sum_{n=0}^{\infty} a_n z^n \quad \& \quad g = \sum_{m=0}^{\infty} b_m z^m$$

If we ever want to consider the product

$$fg = \sum_{n=0}^{\infty} a_n z^n \cdot \sum_{m=0}^{\infty} b_m z^m$$

We simply have to gather like terms and obtain another series of the form

$$\sum_{p=0}^{\infty} c_p z^p$$

It's easy to see that we need c_p to be the sum of all terms such that $n+m=p$. Thus

$$c_p = \sum_{n+m=p} a_n b_m = \sum_{n=0}^{p} a_n b_{p-n}$$

If we'd like to divide series, a similar story follows since we know if g is invertible, then

$$f/g = \frac{\sum_{n=0}^{\infty} a_n z^n}{\sum_{m=0}^{\infty} b_m z^m} = \sum_{p=0}^{\infty} c_p z^p$$

But this means

$$\sum_{n=0}^{\infty} a_n z^n = \sum_{p=0}^{\infty} c_p z^p \cdot \sum_{m=0}^{\infty} b_m z^m$$

Thus

$$a_n = \sum_{p=0}^{n} c_p b_{n-p} = b_0 c_n + \sum_{p=0}^{n-1} c_p b_{n-p} \implies c_n = \frac{1}{b_0}\left(a_n - \sum_{p=0}^{n-1} c_p b_{n-p}\right)$$

This gives us a "nice" formula for computing products and quotients.