Tutorial 4

MAT334 – Complex Variables – Spring 2016

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Solutions

1.4 - # 37,38, 39 Determine whether the series converges

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1+i}{\sqrt{2}}\right)^n, \quad \sum_{n=0}^{\infty} z^n - z^{n-1}, \quad \sum_{k=1}^{\infty} \frac{k^2+i}{(k+i)^4}$$

Solution The first sum converges since we may rewrite

$$\frac{1}{n}\left(\frac{1+i}{\sqrt{2}}\right)^n = \frac{e^{in\pi/4}}{n} = \frac{\cos(n\pi/4)}{n} + i\frac{\cos((n-2)\pi/4)}{n}$$

and we know $\cos(n\pi/4)/n$ converges by the alternating series test in the form

$$\sum_{n=1}^{\infty} \cos(n\pi/4)/n = \frac{1}{\sqrt{2}}\sum_{n=1}^{\infty}(-1)^n\left[\frac{1}{4n-3} + 0 - \frac{1}{4n-1} - \frac{\sqrt{2}}{4n}\right]$$

The second sum is telescoping, thus we know

$$\sum_{n=0}^{N} z^n - z^{n-1} = z^N - \frac{1}{z} \implies \sum_{n=0}^{\infty} z^n - z^{n-1} \quad \begin{cases} \text{converges if } |z| < 1 \text{ or } z = 1 \\ \text{diverges if } |z| \geqslant 1 \text{ or } z = 0 \end{cases}$$

The third sum we may use the limit comparison test with $b_k = 1/k^2$ (since it converges absolutely) , we see

$$\lim_{k\to\infty}\left|\frac{a_k}{b_k}\right| = \lim_{k\to\infty}\left|\frac{1+i/k^4}{(1+i/k)^4}\right| = 1 \implies \sum_{k=1}^{\infty}\frac{k^2+i}{(k+i)^4} \quad \text{converges}$$

□

1.4 - # 41 Suppose that $\sum_{n=1}^{\infty} a_n$ converges. Let $|z| < 1$. Show that the series $\sum_{n=1}^{\infty} a_n z^n$ is convergent.

Solution By the ratio test, we know that $\lim_{n\to\infty}|a_{n+1}/a_n| \leqslant 1$, thus applying the ratio test here shows

$$\lim_{n\to\infty}\left|\frac{a_{n+1}z^{n+1}}{a_n z^n}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right||z| < 1 \implies \text{converges}$$

□

1.6 - # 3 Let γ be the line segment from 2 to $3+i$, compute

$$\int_\gamma |z|^2 dz$$

Solution Define

$$\gamma(t) = 2(1-t) + t(3+i) = 2 + t + it, \quad t \in [0,1]$$

then $\gamma'(t) = 1 + i$, so using the change of variables $z(t) = \gamma(t)$ we see

$$\int_\gamma |z|^2 dz = \int_0^1 [(2+t)^2 + t^2](1+i)dt = (1+i)\left[\frac{(2+t)^3}{3} + \frac{t^3}{3}\right]_0^1 = (1+i)\frac{20}{3}$$

1.6 - # 4 Let γ be a circle of radius 1 centred at -4, oriented counterclockwise, compute

$$\int_\gamma \frac{dz}{z+4}$$

Solution Define

$$\gamma(\theta) = -4 + e^{i\theta} \quad \theta \in [0, 2\pi)$$

then $\gamma'(\theta) = ie^{i\theta}$, so we see

$$\int_\gamma \frac{dz}{z+4} = \int_0^{2\pi} \frac{ie^{i\theta}}{e^{i\theta}} d\theta = i\int_0^{2\pi} d\theta = 2\pi i$$

1.6 - # 9 Show that

$$\frac{1}{2\pi}\int_0^{2\pi} e^{ik\theta}d\theta = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

Then using $\gamma = \{z : |z| = 1\}$ with positive orientation and $D = \{z : |z| < 1\}$, explicitly verify

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-p} = \begin{cases} 1, & p \in D \\ 0, & p \notin \bar{D} \end{cases}$$

Solution We see that when $k \in \mathbb{Z} \setminus \{0\}$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} e^{ik\theta}d\theta = \frac{1}{2\pi}\left[\frac{e^{ik\theta}}{ik}\right]_0^{2\pi} = \frac{e^{2\pi ki} - 1}{2\pi ik} = 0$$

Then when $k = 0$ we see

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta = 1$$

We know that

$$\frac{1}{z-p} = \begin{cases} \sum_{k=0}^\infty \frac{p^k}{z^{k+1}} & |z| = 1, |p| < 1 \\ -\sum_{k=0}^\infty \frac{z^k}{p^{k+1}} & |z| = 1, |p| > 1 \end{cases}$$

by applying the geometric series argument via Partial Sums. Thus if $p \in D$, we see that

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-p} = \frac{1}{2\pi i}\sum_{k=0}^\infty \int_\gamma \frac{p^k}{z^{k+1}}dz = \frac{1}{2\pi i}\sum_{k=0}^\infty p^k \int_0^{2\pi} \frac{ie^{i\theta}d\theta}{e^{i(k+1)\theta}} = \frac{1}{2\pi i}\sum_{k=0}^\infty p^k i \int_0^{2\pi} e^{-ik\theta}d\theta = 1$$

If $p \notin D$, we see that

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-p} = -\frac{1}{2\pi i}\sum_{k=0}^\infty \int_\gamma \frac{z^k}{p^{k+1}}dz = -\frac{1}{2\pi i}\sum_{k=0}^\infty \frac{i}{p^{k+1}}\int_0^{2\pi} e^{i(k+1)\theta}d\theta = 0$$

(Note that we may exchange the integral and the sum by Fubini's Theorem by bounded the sum with $2\pi \max \frac{1}{|z-p|}$)

1.6 - # 13 Use Green's Theorem to derive Green's Identity:

$$\iint_\Omega \nabla u \cdot \nabla v \, dxdy = \int_{\partial\Omega} v \frac{\partial u}{\partial n} \, ds - \iint_\Omega v \Delta u \, dxdy$$

Solution Note that $\partial_n u = \nabla u \cdot n$ and that Green's theorem is

$$\int_\Omega (\nabla \cdot F) dxdy = \int_\Gamma F \cdot nds$$

since $(dy, -dx) = nds$ and $F = (Q, -P)$ to go back to the calculus formulation of Green's. Now if we choose $F = v\nabla u$, the result follows by product rule

$$\int_\Omega (\nabla \cdot v\nabla u) dxdy = \int_\Omega (\nabla v \cdot \nabla u + v \underbrace{\nabla \cdot \nabla}_{=\Delta} u) dxdy = \int_\Gamma v \nabla u \cdot nds = \int_\Gamma v \frac{\partial u}{\partial n} ds$$

Note we may use the complex version of Green's since the regular version implies it by

$$\int_\gamma f(z)dz = \int_\gamma f(x,y)dx + if(x,y)dy = \iint_\Omega \left(i\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} \right) dxdy = i \iint_\Omega \left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} \right) dxdy$$

1.6 - # 15 Let u be a continuous function on $\mathbb{C}$, which is bounded for all z. Let γ_R be the circle $|z| = R$. Show that

$$\lim_{R\to\infty} \int_{\gamma_R} \frac{u(z)}{(z - z_0)^2} dz = 0$$

Solution Since $|u(z)| \leqslant M$ for all $z \in \mathbb{C}$, we have that

$$\left| \int_{\gamma_R} \frac{u(z)}{(z - z_0)^2} dz \right| \leqslant \int_{\gamma_R} \frac{|u(z)|}{|z - z_0|^2} |dz| \leqslant M \int_{\gamma_R} \frac{|dz|}{|z - z_0|^2} \leqslant 2\pi \max_{z \in \{|z|=R\}} \frac{1}{|z - z_0|^2} \leqslant \frac{2\pi}{R - |z_0|}$$

Thus

$$\lim_{R\to\infty} \left| \int_{\gamma_R} \frac{u(z)}{(z - z_0)^2} dz \right| \leqslant \lim_{R\to\infty} \frac{2\pi}{R - |z_0|} = 0$$