Tutorial 2

MAT334 – Complex Variables – Spring 2016

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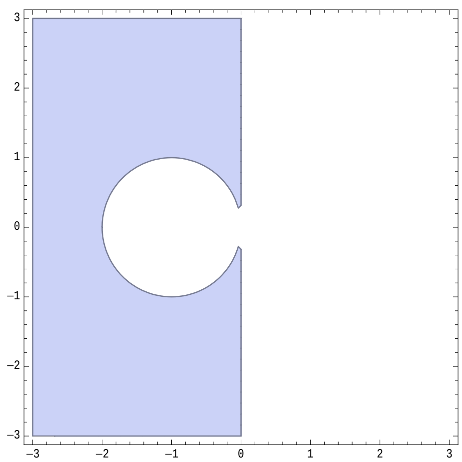
Solutions

1.3 -# 7 Describe the interior and boundary. State whether the set is open or closed or neither . State if the interior of the set is connected (if it has an interior)

$$G = \{z = x + iy : |z + 1| \geqslant 1, x < 0\}$$

Solution The region looks like the dark blue:



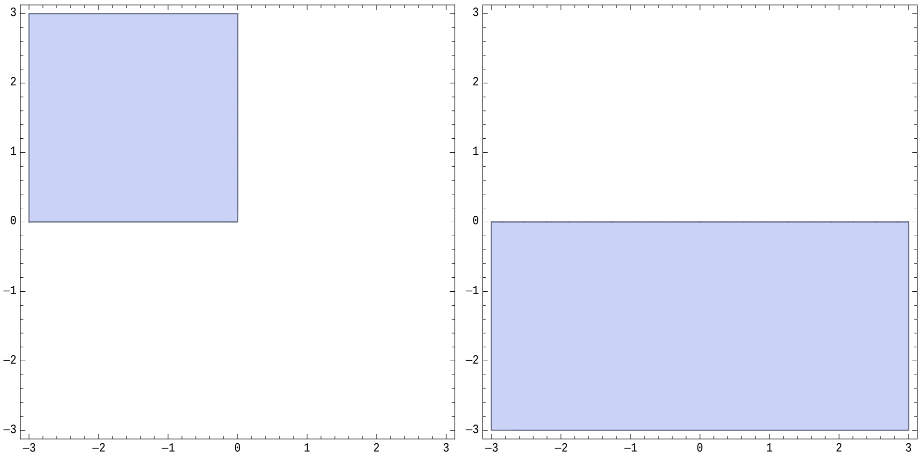
The set is neither open nor closed since points along $x = 0$ are excluded from the set, but the points along the circle are. It is path connected.

1.3 -# 10 Describe the set of points z^2 as z varies over the second quadrant : $\{z = x + iy : x < 0, y > 0\}$. Show that this is an open, connected set.

Solution In polar coordinates we see

$$z^2 = R^2 e^{i2\theta}$$

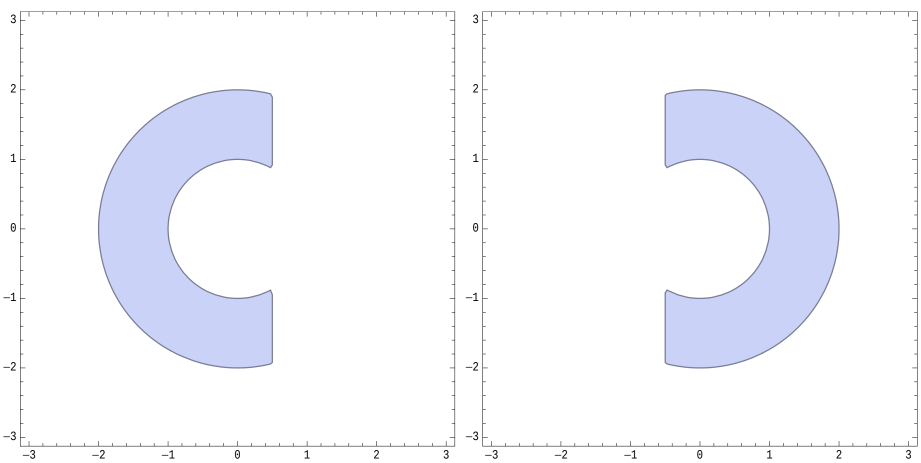
Thus we see the 2nd quadrant gets mapped to the lower half plane, $\{z : Re^{i\theta} : \theta \in (\pi/2, \pi)\} \to \{z : Re^{i\theta} : \theta \in (\pi, 2\pi)\}$ i.e.



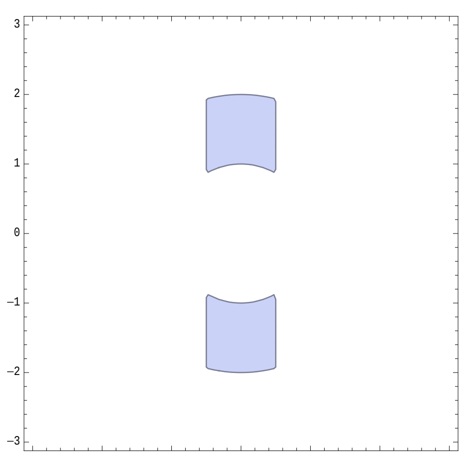
It is path connected since any two points are connected by a line and it is open since for any $z = x + iy$ in the lower half plane we have $B_{y/2}(z) \subseteq \{z : Re^{i\theta} : \theta \in (\pi, 2\pi)\}$

1.3 -# 15 Let $\Omega_1 = \{z : 1 < |z| < 2, \Re z > -1/2\}$ and $\Omega_2 = \{z : 1 < |z| < 2, \Re z < 1/2\}$. Show that both Ω_1 and Ω_2 are domains, but $\Omega_1 \cap \Omega_2$ is not.

Solution It's easy to see that Ω_1 and Ω_2 are both annuli with a "bite" out of them.



The intersection is given by



$\Omega_1 \cap \Omega_2$ isn't a domain since it's not connected. $\square$

1.3 -# 18 An open set D is star-shaped if there is some point p in D with the property that the line segment from p to z lies in D for each z in D. Show that the disc $\{z : |z - z_0| < r\}$ is star-shaped. Show that any convex set is star-shaped.

Solution For the disc, take $p = z_0$, then clearly $\gamma(t) = te^{i\theta} + p$ with $t \in [0, r)$ is a line that connects p to every z in the disc. For a convex set, by definition, we know that for any z_1, z_2 in the set we have $\gamma(t) = tz_1 + (1-t)z_2$ is a line contained in the set. Thus fix $z_1 = p$, and clearly it is star-shaped. $\square$

1.4 - # 4 Find the limit of the sequence if it converges, if it diverges explain why.

$$z_n = \ln\left(1 + \frac{1}{n}\right)$$

Solution By continuity of $\ln x$ for $x > 0$, we have

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} \ln\left(1 + \frac{1}{n}\right) = \ln\left(\lim_{n\to\infty} 1 + \frac{1}{n}\right) = \ln 1 = 0$$

1.4 - # 13 Find the limit of the function at the given point, or explain why it does not exist.

$$f(z) = \frac{|z|^2}{z}, \quad z \neq 0, \quad \text{at } z_0 = 0$$

Solution We already know that $|z|^2 = z\bar{z}$, thus

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{|z|^2}{z} = \lim_{z\to 0} \bar{z} = \lim_{R\to 0} Re^{-i\theta} = 0$$

1.4 - # 18 Find all the points of continuity of the given function

$$g(z) = \frac{1}{(1 - |z|^2)^3}$$

Solution The only issues will come about when the denominator is zero, thus the bad points are found at

$$(1 - |z|^2)^3 = 0 \implies 1 = |z|^2$$

i.e. the circle of radius 1. We conclude the function $g(z)$ is continuous when

$$\{z : |z| > 1\} \quad \text{or} \quad \{z : |z| < 1\}$$

1.4 - # 25 Let f and g be continuous at z_0. Show that $f + g$ and fg are also continuous at z_0. If $g(z_0) \neq 0$, show that $1/g$ is continuous at z_0.

Solution For $(f+g)$, fix ϵ and suppose that when $z \in B_\delta(z_0)$, i.e. $|z - z_0| < \delta$ where $\delta > 0$ we have $|f(z) - f(z_0)| < \epsilon$, same for g. Then by the triangle inequality we see

$$|f(z) + g(z) - f(z_0) - g(z_0)| \leqslant |f(z) - f(z_0)| + |g(z) - g(z_0)| = 2\epsilon$$

Thus $f+g$ is continuous since this may be done for any ϵ. For (fg), fix $\epsilon > 0$, assume the previous continuity assumption. Then

$$\begin{aligned} |f(z)g(z) - f(z_0)g(z_0)| &\leqslant |f(z)(g(z_0)+\epsilon) - f(z_0)g(z_0)| \\ &\leqslant |f(z) - f(z_0)||g(z_0)| + \epsilon|f(z)| \\ &\leqslant \epsilon(|f(z_0)| + |g(z_0)|) + \epsilon^2 \leqslant \tilde{\epsilon} \end{aligned}$$

Thus fg is continuous since this may be done for any $\tilde{\epsilon}$. We sketch the inverted function case. Assume $g(z_0) \neq 0$, then we may write

$$\left|\frac{1}{g(z)} - \frac{1}{g(z_0)}\right| = \left|\frac{g(z) - g(z_0)}{g(z)g(z_0)}\right| \leqslant C(\epsilon)|g(z) - g(z_0)|$$

As long as $g(z) \neq 0$ on $B_\delta(z_0)$, which is given since g is continuous.

1.4 - # 32 Determine whether the given series converges or diverges.

$$\sum_{n=1}^{\infty} n\left(\frac{1}{2i}\right)^n$$

Solution By the ratio test, we see that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n+1}{n}\left|\frac{1}{2i}\right| = \lim_{n\to\infty}\frac{1}{2} + \frac{1}{2n} = \frac{1}{2} < 1$$

Thus the test concludes the series converges. □