Tutorial 10

MAT334 – Complex Variables – Spring 2016

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SOLUTIONS

3.1 - # 5 Determine the number of zeros in the first quadrant.

$$f(z) = z^9 + 5z^2 + 3$$

Solution Recall the argument principal,

$$\frac{1}{2\pi}\Delta \arg f(z)\Big|_{\gamma} = \#N - \#P$$

where $\#N$ and $\#P$ are the number of zeros and poles inside γ. Now consider the quarter disc of radius R, and let's check what happens to the $f(z)$ as it moves around this contour. On the real axis, we see $z = x$ and

$$f(x) = x^9 + 5x^2 + 3 \geqslant 3 \quad \text{when } x \geqslant 0 \implies \Delta \arg f(x)\Big|_{\gamma_x} = 0$$

On the arc, we have $z = Re^{i\theta}$ with $\theta \in [0, \pi/2]$, so

$$f(Re^{i\theta}) = R^9\left(e^{9i\theta} + \frac{5e^{2i\theta}}{R^7} + \frac{3}{R^9}\right)$$

Thus as R gets large, we the leading term dominating. So we see

$$\Delta \arg f(Re^{i\theta})\Big|_{\gamma_R} = 9 * \frac{\pi}{2}$$

On the y axis, we have $z = iy$ and

$$f(iy) = iy^9 - 5y^2 + 3$$

we see that $y = R$ lives in the 2nd quadrant and pulls towards the imaginary axis in the limit, then moves back to the first quadrant's real axis as $y \to 0$. So

$$\Delta \arg f(iy)\Big|_{\gamma_y} = -\frac{\pi}{2}$$

Thus the argument principal tells us

$$\#N - \underbrace{\#P}_{=0} = \frac{1}{2\pi}\Delta \arg f(z)\Big|_{\gamma} = \frac{\Delta \arg f(x)\Big|_{\gamma_x} + \Delta \arg f(Re^{i\theta})\Big|_{\gamma_R} + \Delta \arg f(iy)\Big|_{\gamma_y}}{2\pi} = 2$$

i.e. there are 2 zeros in the 1st quadrant.

3.1 - # 8 Determine the number of zeros in the upper half-plane

$$f(z) = 2z^4 - 2iz^3 + z^2 + 2iz - 1$$

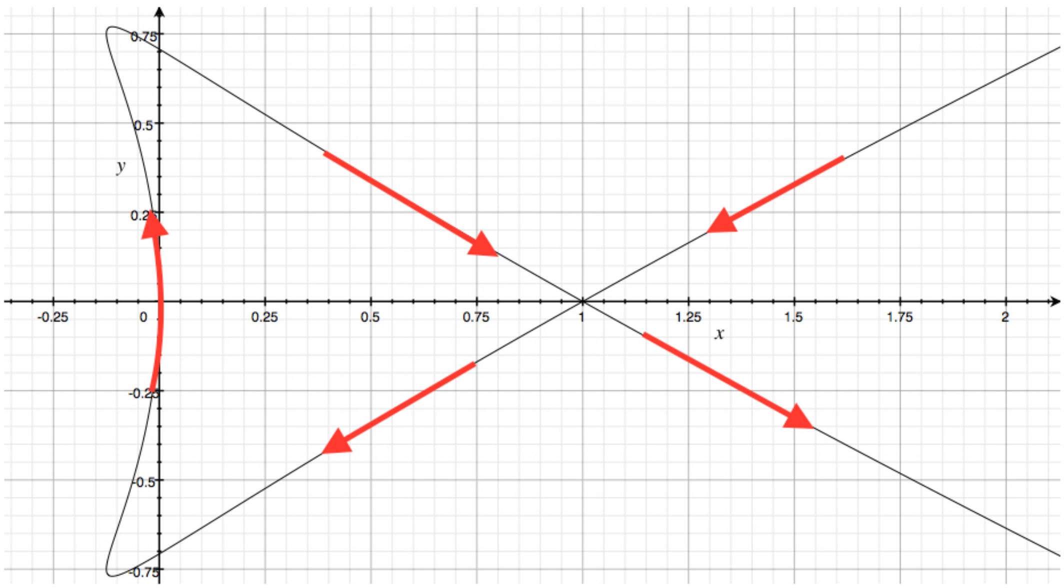
Solution Consider the contour of a half disc with radius R. On the arc, we see that $z = Re^{i\theta}$ with $\theta \in [0, \pi]$. Where the leading order of the f is given by

$$f(Re^{i\theta}) = R^4 \left(2e^{4i\theta} + \mathcal{O}\left(\frac{1}{R}\right)\right) \implies \Delta \arg f(Re^{i\theta})\Big|_{\gamma_R} = 4\pi$$

On the real axis, we see

$$f(x) = 2x^4 + x^2 - 1 - 2ix(x^2 - 1)$$

We see the important points to take note of are the zeros, which happen at $x = \pm 1, 0$ for the imaginary part, and at $x = (-1 \pm 3)/4$ for the real part. When $x < -1$, we see that $\Re f > 0$ and $\Im f > 0$ (i.e. we're in the first quad). Next we see $-1 < x < (-1-3)/4$ has $\Re f > 0$ but $\Im < 0$, so we've moved into the fourth quadrant. If we continue checking, we find the curve looks something like



Thus we see that (noting the pull to the real axis in the limit)

$$\Delta \arg f(x)\Big|_{\gamma_x} = -2\pi$$

so we have that

$$\#N - \underbrace{\#P}_{=0} = \frac{1}{2\pi}\Delta \arg f(z)\Big|_{\gamma} = \frac{\Delta \arg f(x)\Big|_{\gamma_x} + \Delta \arg f(Re^{i\theta})\Big|_{\gamma_R}}{2\pi} = 1$$

□

3.1 - #14 Determine the number of zeros in the annulus $0 < |z| < 2$

$$f(z) = ze^z - \frac{1}{4}$$

Solution Consider $g(z) = -ze^z$ on $\gamma = \{z : |z| = 2\}$, then

$$|f(z) + g(z)| = \frac{1}{4} < 2e^{\Re z} = |ze^z| = |g(z)|$$

Thus Rouché's Theorem tells us that f and g have the same number of zero's. We see that $g(z)$ only has 1 zero at $z = 0$, thus f has 1 zero. □

3.1 - #20 Suppose that f is analytic on a domain containing $\{z : |z| \leqslant 1\}$ and that $|f(e^{i\theta})| < 1$, $0 \leqslant \theta \leqslant 2\pi$. Show that f has exactly one fixed point in the disc $|z| < 1$, i.e. show $f(z) = z$ has only one solution in the unit disc.

Solution Consider $g(z) = f(z) - z$ and $h(z) = z$ on $\{z : |z| = 1\}$

$$|g(z) + h(z)| = |f(z)| < 1 = |z| \quad \text{on } \{z : |z| = 1\}$$

Thus Rouché's Theorem tells us that $f(z) - z$ and z have the same number of zero's in the unit disc. Since z only has 1 zero at $z = 0$, we see there is only one point z_0, such that $f(z_0) = z_0$ □

3.2 - #3 Find the maximum value of $|g(z)|$,

$$g(z) = \frac{z}{4z^2 - 1}$$

as z varies over the region $\Omega = \{z : |z| \geqslant 1\}$.

Solution By the maximum modulus principal, we know that the max is obtained on the boundary of the domain, i.e. $|z| = 1$, thus we see

$$\max_{z\in\Omega} |g(z)|^2 = \max_{z\in\Omega} \left|\frac{z}{4z^2 - 1}\right|^2 = \max_{|z|=1} \frac{|z|^2}{(4z^2 - 1)(4\bar{z}^2 - 1)} = \max_{|z|=1} \frac{|z|^2}{16|z|^4 - 4(z^2 + \bar{z}^2) + 1} = \max_{\theta\in[0,2\pi]} \frac{1}{17 - 8\cos(2\theta)}$$

Now it's easy to see that $\theta = 0$ or π produces the largest modulus, thus

$$\max_{z\in\Omega} |g(z)| = \frac{1}{3}$$

and it occurs at $z = \pm 1$.

□

3.2 -# 9 Suppose that f is analytic on a domain D, which contains a simple closed curve γ and the inside of γ. If $|f|$ is constant on γ, then either f is constant or f has a zero inside γ.

Solution The maximum modulus principal states that an analytic function must take the maximum on the boundary of the domain. If f has no zero in the D, apply the maximum principal to $1/f$, which is analytic on the D. The only way for $1/f$ and f to have a max on entire boundary, is if $f(z) = \mathit{const}$. If f has a zero, everything checks out since $1/f$ isn't analytic.