Test 3

MAT334 – Complex Variables – Spring 2016

Christopher J. Adkins

SOLUTIONS

Question 1 Compute

$$\oint_{|z|=2} \left[\frac{z^2}{z-1}\right]^{10} dz$$

Solution Recall Cauchy's Integral formula; If $f(z)$ is analytic in the interior of γ,where γ is a simple closed curve, then for z in the interior of γ

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_{\gamma} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

In the above question, we see that $1 \in \{z : |z| < 2\} = \text{int}(\gamma)$, so

$$\oint_{|z|=2} \left[\frac{z^2}{z-1}\right]^{10} dz = \oint_{|z|=2} \frac{z^{20}}{(z-1)^{10}} dz = \frac{2\pi i}{9!} f^{(9)}(1)$$

where $f(z) = z^{20}$(which is clearly analytic). Computing the derivative shows

$$\oint_{|z|=2} \left[\frac{z^2}{z-1}\right]^{10} dz = 2\pi i \frac{20!}{9!11!} = 2\pi i \binom{20}{9}$$

□

Question 2 Let

$$f(z) = \frac{1}{z^4 + 1}$$

- Determine the residues of f at the poles which are in the upper half plane, $\{\Im(z) > 0\}$.
- Compute the indefinite integral

$$\int_{-\infty}^{\infty} \frac{dx}{x^4 + 1}$$

Solution The poles of the function are given by the zeros of the denominator, we see

$$z^4 + 1 = 0 \implies z^4 = -1 = e^{i\pi + 2\pi n} \implies z_n = e^{i\pi/4 + \pi n/2}, \quad n = 0, 1, 2, 3$$

For differences of notation, note

$$z_0 = e^{i\pi/4} = \frac{1+i}{\sqrt{2}} \quad \& \quad z_1 = e^{i3\pi/4} = \frac{-1+i}{\sqrt{2}} \quad \& \quad z_2 = e^{i5\pi/4} = \frac{-1-i}{\sqrt{2}} \quad \& \quad z_3 = e^{i7\pi/4} = \frac{1-i}{\sqrt{2}}$$

Clearly $n = 0, 1$ are the poles in the upper half plane. Note that all poles are simple, thus we may apply the formula

$$\text{Res}\left(\frac{P(z)}{Q(z)}; z_0\right) = \frac{P(z_0)}{Q'(z_0)}$$

We see

$$\text{Res}\,(f, z_0) = \frac{1}{4z_0^3} = \frac{z_0}{4z_0^4} = -\frac{1+i}{4\sqrt{2}}$$
$$\text{Res}\,(f, z_1) = \frac{1}{4z_1^3} = \frac{z_1}{4z_1^4} = -\frac{-1+i}{4\sqrt{2}}$$

To compute the integral, choose the contour of a half disk of radius R with base on the x-axis. Then by linearity, we have by the residue theorem that

$$\int_\gamma f(z)dz = \int_{\gamma_x} f(z)dz + \int_{\gamma_{arc}} f(z)dz = 2\pi i \times \sum_{z_i \in \text{int}(\gamma)} \text{Res}\,(f(z); z_i) = 2\pi i\,(\text{Res}\,(f, z_0) + \text{Res}\,(f, z_1)) = \frac{\pi}{\sqrt{2}}$$

Note that the contribution from γ_{arc} dies in the limit as $R \to \infty$ since

$$\left|\int_{\gamma_{arc}} f(z)dz\right| \leqslant \frac{const}{R^3} \to 0$$

Thus

$$\lim_{R\to\infty} \int_{\gamma_x} f(z)dz = \int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}$$

□

Question 3 Let

$$f(z) = \frac{1+z}{1-\cos(z)}$$

- Determine the order of the pole at $z_0 = 2\pi$
- Determine the first 3 coefficients of the Laurent expansion at $z_0 = 2\pi$

Solution Noting that $\cos(z)$ is 2π periodic and recalling the series expansion of cosine gives

$$\cos(z) = \cos(z - 2\pi) = \sum_{n\geqslant 0} (-1)^n \frac{(z-2\pi)^{2n}}{(2n)!} = 1 - \frac{(z-2\pi)^2}{2} + \frac{(z-2\pi)^4}{24} - \mathcal{O}((z-2\pi)^6)$$

Thus, locally around $z = 2\pi$ we see

$$f(z) = \frac{1+z}{1-\cos(z)} \approx 2\frac{1+2\pi}{(z-2\pi)^2}$$

which shows that $f(z)$ has a pole of order 2 at $z_0 = 2\pi$. Thus, we know the Laurent expansion of $f(z)$ around $z_0 = 2\pi$ must take the form

$$f(z) = \sum_{n\geqslant -2} a_n(z-z_0)^n = \frac{a_{-2}}{(z-2\pi)^2} + \frac{a_{-1}}{z-2\pi} + a_0 + \mathcal{O}(z-2\pi)$$

Setting both expressions of $f(z)$ equal to one another gives us a recursive method to find the coefficients through series multiplication. Indeed, we see

$$\frac{1+z}{1-\cos(z)} = \sum_{n\geqslant -2} a_n(z-z_0)^n$$

Thus (note the shift in the LHS for ease of matching coefficients in the next step)

$$\begin{aligned}1+2\pi+(z-2\pi) =&(1-\cos(z))\sum_{n\geqslant -2} a_n(z-z_0)^n\\ =&\left(\frac{(z-2\pi)^2}{2}-\frac{(z-2\pi)^4}{24}\right)\left(\frac{a_{-2}}{(z-2\pi)^2}+\frac{a_{-1}}{z-2\pi}+a_0\right)+\mathcal{O}((z-2\pi)^3)\\ =&\frac{a_{-2}}{2}+\frac{a_{-1}}{2}(z-2\pi)+\left(\frac{a_0}{2}-\frac{a_{-2}}{24}\right)(z-2\pi)^2+\mathcal{O}((z-2\pi)^3)\end{aligned}$$

By comparing the LHS and the RHS, we read off

$$\begin{cases}\frac{a_{-2}}{2}=1+2\pi\\ \frac{a_{-1}}{2}=1\\ \frac{a_0}{2}-\frac{a_{-2}}{24}=0\end{cases} \implies \begin{cases}a_{-2}=2(1+2\pi)\\ a_{-1}=2\\ a_0=\frac{1+2\pi}{6}\end{cases}$$

i.e.

$$\frac{1+z}{1-\cos(z)}=2\frac{1+2\pi}{(z-2\pi)^2}+\frac{2}{z-2\pi}+\frac{1+2\pi}{6}+\mathcal{O}(z-2\pi)$$

□

Question 4 Compute

$$\oint_{|z|=1}\frac{\sin^2(z)}{1-\cos(z)}dz$$

Solution Note that the integrand may be extended to (using the Pythagorean Theorem)

$$f(z)=\frac{\sin^2(z)}{1-\cos(z)}=\frac{1-\cos^2(z)}{1-\cos(z)}=\frac{(1-\cos(z))(1+\cos(z))}{1-\cos(z)}=1+\cos(z)$$

which is an entire function. Cauchy's Theorem tells us the integral, over a closed curve, of a function that is analytic in the interior is identically zero. Thus

$$\oint_{|z|=1}\frac{\sin^2(z)}{1-\cos(z)}dz=0$$

□